Metatheory of LF Extended with Dependent Pair and Unit Types

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Abstract

We study an extension of the well known LF type theory with dependent pair and unit types. The important metatheoretic properties proved in this study are decidability of type checking and the existence of canonical forms. The study follows standard techniques introduced by Harper and Pfenning [5] together with extensions studied by Vanderwaart and Crary [7].

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1 Introduction

The dependently typed lambda calculus LF was introduced by Harper, Honsell and Plotkin [4] to serve as a logical framework. This language has since been used extensively to represent logics. This calculus can be described as a dependently-typed lambda calculus, with three levels: objects, type families and kinds. Both type families and kinds can depend on objects.

Our goal is to extend a functional language such as ML to be able to talk statically about logics represented in LF [6]. One key problem in this scenario is to deal with open terms. We have argued in earlier work [6] that we need only represent closed terms if we reify contexts within the language, and reify open terms as terms abstracted over the reified contexts. Our representation of contexts is as products of the types in the context, in a manner similar to the "telescopes" of deBruijn [3]. Since terms in the context can depend on terms abstracted over earlier in the context, we in fact require dependent pair types, also known as Σ-types. The null context is reified as the unit type, with a single canonical inhabitant.

In this paper, we will take a first step towards such a language. We will concentrate on the metatheory of the extended LF sketched out above. The current work studies LF extended with Σ types and unit, which we shall call $\mathrm{LF}^{\Sigma,1}$. We will also need family-level abstractions to abstract over contexts. This was present in the original proposal for LF [4], but was dropped in the study of its metatheory [5]. We will add them back to the calculus.

A crucial property required of a logical framework is the existence of canonical forms. The "adequacy theorem" which says that the representation is meaningful establishes a bijection between the propositions and proofs in the represented logic and canonical forms of certain types. Canonical forms are taken to be β-normal, η-long forms. A second important property we would like a logical framework to have is decidability of checking proofs. In LF, this reduces to being able to decide type checking, since proofs are represented as terms and judgments as types.

These properties have been studied before for related languages. We will summarize the previous contributions briefly. The first presentation of LF [4] relied on β conversion as the notion of definitional equality, leaving out the technically complex η conversion entirely. This posed a problem with the method sketched out above, since canonical forms are η-long forms. Subsequently, Harper and Pfenning [5] reformulated LF to make the definitional equality a typed judgment form instead of an untyped reduction scheme. This definitional equality compared objects at a specified type and in a specified context, and axiomatized a congruence relation generated by the $\beta\eta$ convertibility of well typed terms. With this innovation, a new algorithm to decide type checking could be produced. The key component of a type-checking algorithm is an algorithm to decide definitional equality. They produced such an algorithm which is directed by the types at which objects are being compared. The algorithm works similarly to one by Coquand [1], except that comparisons is directed by types and not by the shape of terms. The novel type-directed algorithm was proved correct using the so-called method of logical relations. For technical reasons, Harper and Pfenning did not treat abstraction at the level of type families. Later, Vanderwaart and Crary [7] extended Harper and Pfenning's method to the Linear Logical Framework LLF. They provided a separate argument to prove a necessary property of family level abstractions in this setting.

We will follow Harper and Pfenning [5] and Vanderwaart and Crary [7], and extend their equality algorithm to our language. As mentioned already, this algorithm is directed by the type at which objects are being compared. To avoid technical problems with proofs, it erases all dependencies from the type at which the comparison is done. The algorithm itself is simple. The main bulk of the work is to prove this algorithm sound and complete with respect to definitional equality. Completeness requires a complex proof based on a Kripke logical relation to make the inductive case go through. This method is described for a much simpler language by Crary [2].

Our study is a rather direct extension of Vanderwaart and Crary's work, extended with a dependent pair and a unit type. For readers familiar with that work, our sequence of lemmas and theorems will be familiar. Our work also differs in the minor way that canonical forms is proved directly for terms in the language. Harper and Pfenning define a related but different notion called quasi-canonical forms, which do not syntactically belong to the language. Vanderwaart and Crary defer to their work, and do not prove the canonical forms property.

1.1 Overview

In section 2 we present the abstract syntax of our language. We define the typing and definitional equality judgments in section 3. Some elementary structural properties we require are proved in section 4. In particular, we will need regularity, *i.e.* terms appearing in typing judgments are well-formed. We will also need a property called functionality of substitutions, *i.e.* equal substitutions map equal arguments to equal arguments. Another property we need is called injectivity, which says that equal products and sums have equal components. This property is impossible to prove by simple induction on the equality judgment in the presence of the non-trivial equality induced by the presence of family-level abstractions. We follow Vanderwaart and Crary in proving this by means of a logical relation in section 5.

Next, we produce an algorithm in section 6 for deciding definitional equality. The completeness of this algorithm with respect to definitional equality is proved in section 7. We prove both soundness and existence of canonical forms together, in section 8. Finally in section 9 we extend the algorithm for equality to a sound and complete algorithm for typechecking. We prove decidability for all judgment forms.

2 Abstract Syntax

We start with presenting the abstract syntax of our language. This is generated by the following grammar.

Kinds	$\mathsf{K} ::=$	type	kind of types
	$\mid$	$\Pi\mathsf{x}{:}\mathsf{A}.\mathsf{K}$	dependent product kind
Families	$\mathsf{A} ::=$	a	family constants
	$\mid$	$\lambda\mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2$	family level abstraction
	$\mid$	$\mathsf{A}\,\mathsf{M}$	family application
	$\mid$	$\Pi\mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2$	family of functions
	$\mid$	$\Sigma\mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2$	family of products
	$\mid$	$\mathsf{1}$	unit type
Objects	$\mathsf{M} ::=$	c	object constants
	$\mid$	x	object variables
	$\mid$	$\lambda\mathsf{x}{:}\mathsf{A}.\mathsf{M}$	object functions
	$\mid$	$\mathsf{M}_1\,\mathsf{M}_2$	object level application
	$\mid$	$\langle\mathsf{M}_1,\mathsf{M}_2\rangle^{\mathsf{A}}$	pairs of objects
	$\mid$	$\pi_i\mathsf{M}\quad(i=1,2)$	projections from pairs
	$\mid$	$\langle\rangle$	unit object
Signatures	$\Sigma ::=$	$\cdot$	empty
	$\mid$	$\Sigma,\mathsf{a}{:}\mathsf{K}$	extension by family level constant
	$\mid$	$\Sigma,\mathsf{c}{:}\mathsf{A}$	extension by object level constant
Contexts	$\Gamma ::=$	$\cdot$	empty
	$\mid$	$\Gamma,\mathsf{x}{:}\mathsf{A}$	context extension

There are three levels of terms: objects, type families and kinds. Kinds classify families, and families belonging to kind type are called *types*. Types classify objects. The family level includes Π and Σ types, a unit type, as well as family-level abstractions and applications. We have object and family-level constants. A signature records the types and kinds assigned to these constants. We have object-level variables, and these are provided types by contexts. The other object level constructs are abstractions and applications, pairs and projections, and a canonical inhabitant of the unit type. Contexts, which assign types to variables, can be assumed to be ordered lists. Ordering is important because of dependencies. We assume that a variable is not bound more than once in contexts. This can always be assured in the standard way in all judgments.

We define a partial order on contexts syntactically. $\Gamma_1 \subseteq \Gamma_2$ holds if $\Gamma_1(\mathsf{x}) = \Gamma_2(\mathsf{x})$ for every $\mathsf{x} \in \mathsf{Dom}(\Gamma_1)$. Thus if $\Gamma_1 \subseteq \Gamma_2$ then $\mathsf{Dom}(\Gamma_1) \subseteq \mathsf{Dom}(\Gamma_2)$, and Γ_1 appears as a subsequence of Γ_2.

Substitutions	$\sigma ::=$	$\cdot$	empty
	$\mid$	$\sigma,\mathsf{M}/\mathsf{x}$	cons

Substitutions are finite maps from object variables to objects. They are defined as substituting for all variables in their domain simultaneously. We assume all object variables occurring in substitutions are distinct. We write id_Γ for the substitution which is identity on all variables in the domain of Γ. The result of applying substitutions on objects, families and kinds is written as $\mathsf{M}[\sigma]$, $\mathsf{A}[\sigma]$ and $\mathsf{K}[\sigma]$, and this notation is extended to all judgments $\mathcal{J}$ of the theory.

3 Static Semantics

3.1 Judgment Forms

The static semantics is presented in the form of 8 mutually recursive judgments, whose meanings are explained below.

$\vdash \Sigma : \mathsf{sig}$	Σ is a valid signature
$\vdash \Gamma : \mathsf{ctx}$	Γ is a valid context
$\Gamma \vdash \mathsf{M} : \mathsf{A}$	M has type A
$\Gamma \vdash \mathsf{A} : \mathsf{K}$	A has kind K
$\Gamma \vdash \mathsf{K} : \mathsf{kind}$	K is a valid kind
$\Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : \mathsf{A}$	M_1 equals M_2 at type A
$\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_2 : \mathsf{K}$	A_1 equals A_2 at kind K
$\Gamma \vdash \mathsf{K}_1 = \mathsf{K}_2 : \mathsf{kind}$	K_1 equals K_2

The last three judgments define a typed definitional equality judgment. These equate terms at a particular type, families at particular kinds, and kinds.

3.2 Inference Rules

Recall that signatures assign types and kinds to constants. The first judgment ensures that such types and kinds are well-formed.

$\boxed{\vdash \Sigma : \mathsf{sig}}$

Empty

$$\frac{}{\vdash \cdot : \mathsf{sig}}$$

Family Constant

$$\frac{\vdash \Sigma : \mathsf{sig} \qquad \cdot \vdash_\Sigma \mathsf{K} : \mathsf{kind}}{\vdash \Sigma, \mathsf{a}{:}\mathsf{K} : \mathsf{sig}}$$

Object Constant

$$\frac{\vdash \Sigma : \mathsf{sig} \qquad \cdot \vdash_\Sigma \mathsf{A} : \mathsf{type}}{\vdash \Sigma, \mathsf{c}{:}\mathsf{A} : \mathsf{sig}}$$

From now on we assume fixed a valid signature Σ and omit it from the judgments. All further judgments assume this signature to be fixed.

$\boxed{\vdash \Gamma : \mathsf{ctx}}$

Empty

$$\frac{}{\vdash \cdot : \mathsf{ctx}}$$

Cons

$$\frac{\vdash \Gamma : \mathsf{ctx} \qquad \Gamma \vdash \mathsf{A} : \mathsf{type}}{\vdash \Gamma, \mathsf{x}{:}\mathsf{A} : \mathsf{ctx}}$$

$\boxed{\Gamma \vdash \mathsf{M} : \mathsf{A}}$

Variables

$$\frac{\Gamma(\mathsf{x}) = \mathsf{A}}{\Gamma \vdash \mathsf{x} : \mathsf{A}}$$

Constants

$$\frac{\Sigma(\mathsf{c}) = \mathsf{A}}{\Gamma \vdash \mathsf{c} : \mathsf{A}}$$

Applications

$$\frac{\Gamma \vdash \mathsf{M}_1 : \Pi \mathsf{x}{:}\mathsf{A}_2.\mathsf{A}_1 \qquad \Gamma \vdash \mathsf{M}_2 : \mathsf{A}_2}{\Gamma \vdash \mathsf{M}_1\, \mathsf{M}_2 : \mathsf{A}_1\, [\mathsf{M}_2/\mathsf{x}]}$$

Abstractions

$$\frac{\Gamma \vdash \mathsf{A}_1 : \mathsf{type} \qquad \Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{M}_2 : \mathsf{A}_2}{\Gamma \vdash \lambda \mathsf{x}{:}\mathsf{A}_1.\mathsf{M}_2 : \Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2}$$

Pairs

$$\frac{\Gamma \vdash \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 : \mathsf{type} \qquad \Gamma \vdash \mathsf{M}_1 : \mathsf{A}_1 \qquad \Gamma \vdash \mathsf{M}_2 : \mathsf{A}_2\, [\mathsf{M}_1/\mathsf{x}]}{\Gamma \vdash \langle \mathsf{M}_1, \mathsf{M}_2 \rangle^{\Sigma \mathsf{x}:\mathsf{A}_1.\mathsf{A}_2} : \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2}$$

Projections

$$\frac{\Gamma \vdash \mathsf{M} : \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2}{\Gamma \vdash \pi_1 \mathsf{M} : \mathsf{A}_1}$$

$$\frac{\Gamma \vdash \mathsf{M} : \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2}{\Gamma \vdash \pi_2 \mathsf{M} : \mathsf{A}_2\, [\pi_1 \mathsf{M}/\mathsf{x}]}$$

Unit

$$\frac{}{\Gamma \vdash \langle\rangle : 1}$$

Type Conversion

$$\frac{\Gamma \vdash \mathsf{M} : \mathsf{A}_1 \qquad \Gamma \vdash \mathsf{A}_1 = \mathsf{A}_2 : \mathsf{type}}{\Gamma \vdash \mathsf{M} : \mathsf{A}_2}$$

$\boxed{\Gamma \vdash \mathsf{A} : \mathsf{K}}$

Constants

$$\frac{\Sigma(\mathsf{a}) = \mathsf{K}}{\Gamma \vdash \mathsf{a} : \mathsf{K}}$$

Abstractions

$$\frac{\Gamma \vdash \mathsf{A}_1 : \mathsf{type} \quad \Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{A}_2 : \mathsf{K}}{\Gamma \vdash \lambda \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 : \Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{K}}$$

Applications

$$\frac{\Gamma \vdash \mathsf{A}_1 : \Pi \mathsf{x}{:}\mathsf{A}_2.\mathsf{K} \qquad \Gamma \vdash \mathsf{M} : \mathsf{A}_2}{\Gamma \vdash \mathsf{A}_1\, \mathsf{M} : \mathsf{K}\, [\mathsf{M}/\mathsf{x}]}$$

Dependent Function Types

$$\frac{\Gamma \vdash \mathsf{A}_1 : \mathsf{type} \qquad \Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{A}_2 : \mathsf{type}}{\Gamma \vdash \Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 : \mathsf{type}}$$

Dependent Pair Types

$$\frac{\Gamma \vdash \mathsf{A}_1 : \mathsf{type} \qquad \Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{A}_2 : \mathsf{type}}{\Gamma \vdash \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 : \mathsf{type}}$$

Unit

$$\frac{}{\Gamma \vdash \mathsf{1} : \mathsf{type}}$$

Kind Conversion

$$\frac{\Gamma \vdash \mathsf{A} : \mathsf{K}_1 \qquad \Gamma \vdash \mathsf{K}_1 = \mathsf{K}_2 : \mathsf{kind}}{\Gamma \vdash \mathsf{A} : \mathsf{K}_2}$$

$\boxed{\Gamma \vdash \mathsf{K} : \mathsf{kind}}$

Type

$$\frac{}{\Gamma \vdash \mathsf{type} : \mathsf{kind}}$$

Products

$$\frac{\Gamma \vdash \mathsf{A} : \mathsf{type} \qquad \Gamma, \mathsf{x}{:}\mathsf{A} \vdash \mathsf{K} : \mathsf{kind}}{\Gamma \vdash \Pi \mathsf{x}{:}\mathsf{A}.\mathsf{K} : \mathsf{kind}}$$

Definitional equality is presented in the form of the three judgments detailed below. These axiomatize equality between objects, type families and kinds.

$\boxed{\Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : \mathsf{A}}$

Variables

$$\frac{\Gamma(\mathsf{x}) = \mathsf{A}}{\Gamma \vdash \mathsf{x} = \mathsf{x} : \mathsf{A}}$$

Constants

$$\frac{\Sigma(\mathsf{c}) = \mathsf{A}}{\Gamma \vdash \mathsf{c} = \mathsf{c} : \mathsf{A}}$$

Applications

$$\frac{\Gamma \vdash \mathsf{M}_{11} = \mathsf{M}_{21} : \Pi \mathsf{x}{:}\mathsf{A}_2.\mathsf{A}_1 \qquad \Gamma \vdash \mathsf{M}_{12} = \mathsf{M}_{22} : \mathsf{A}_2}{\Gamma \vdash \mathsf{M}_{11}\,\mathsf{M}_{12} = \mathsf{M}_{21}\,\mathsf{M}_{22} : \mathsf{A}_1\,[\mathsf{M}_{12}/\mathsf{x}]}$$

Abstractions

$$\frac{\Gamma \vdash \mathsf{A}_{11} = \mathsf{A}_1 : \mathsf{type} \qquad \Gamma \vdash \mathsf{A}_{12} = \mathsf{A}_1 : \mathsf{type} \qquad \Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{M}_1 = \mathsf{M}_2 : \mathsf{A}_2}{\Gamma \vdash \lambda \mathsf{x}{:}\mathsf{A}_{11}.\mathsf{M}_1 = \lambda \mathsf{x}{:}\mathsf{A}_{12}.\mathsf{M}_2 : \Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2}$$

Extensionality
at Π type

$$\frac{\Gamma \vdash \mathsf{A}_1 : \mathsf{type} \quad \Gamma \vdash \mathsf{M}_1 : \Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 \quad \Gamma \vdash \mathsf{M}_2 : \Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 \quad \Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{M}_1\,\mathsf{x} = \mathsf{M}_2\,\mathsf{x} : \mathsf{A}_2}{\Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : \Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2}$$

Parallel β-conversion
for function application

$$\frac{\Gamma \vdash \mathsf{A}_1 : \mathsf{type} \qquad \Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{M}_{12} = \mathsf{M}_{22} : \mathsf{A}_2 \qquad \Gamma \vdash \mathsf{M}_{11} = \mathsf{M}_{21} : \mathsf{A}_1}{\Gamma \vdash (\lambda \mathsf{x}{:}\mathsf{A}_1.\mathsf{M}_{12})\,\mathsf{M}_{11} = \mathsf{M}_{22}\,[\mathsf{M}_{21}/\mathsf{x}] : \mathsf{A}_2\,[\mathsf{M}_{11}/\mathsf{x}]}$$

Pairs

$$\frac{\Gamma \vdash \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 : \mathsf{type} \qquad \Gamma \vdash \mathsf{M}_{11} = \mathsf{M}_{21} : \mathsf{A}_1 \qquad \Gamma \vdash \mathsf{M}_{12} = \mathsf{M}_{22} : \mathsf{A}_2\,[\mathsf{M}_{11}/\mathsf{x}]}{\Gamma \vdash \langle \mathsf{M}_{11}, \mathsf{M}_{12}\rangle^{\Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2} = \langle \mathsf{M}_{21}, \mathsf{M}_{22}\rangle^{\Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2} : \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2}$$

Projections

$$\frac{\Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2}{\Gamma \vdash \pi_1 \mathsf{M}_1 = \pi_1 \mathsf{M}_2 : \mathsf{A}_1}$$

$$\frac{\Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2}{\Gamma \vdash \pi_2 \mathsf{M}_1 = \pi_2 \mathsf{M}_2 : \mathsf{A}_2\,[\pi_1 \mathsf{M}_1/\mathsf{x}]}$$

Extensionality
at Unit Type

$$\frac{\Gamma \vdash \mathsf{M}_1 : 1 \quad \Gamma \vdash \mathsf{M}_2 : 1}{\Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : 1}$$

Parallel β-Conversion
for pair projection

$$\frac{\Gamma \vdash \mathsf{M}_1 = \mathsf{M}_3 : \mathsf{A}_1 \qquad \Gamma \vdash \mathsf{M}_2 : \mathsf{A}_2}{\Gamma \vdash \pi_1 \langle \mathsf{M}_1, \mathsf{M}_2 \rangle^{\mathsf{A}} = \mathsf{M}_3 : \mathsf{A}_1}$$

$$\frac{\Gamma \vdash \mathsf{M}_1 : \mathsf{A}_1 \qquad \Gamma \vdash \mathsf{M}_2 = \mathsf{M}_3 : \mathsf{A}_2}{\Gamma \vdash \pi_2 \langle \mathsf{M}_1, \mathsf{M}_2 \rangle^{\mathsf{A}} = \mathsf{M}_3 : \mathsf{A}_2}$$

Extensionality
at Σ type

$$\frac{\begin{array}{c}\Gamma \vdash \mathsf{M}_1 : \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 \quad \Gamma \vdash \mathsf{M}_2 : \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 \\ \Gamma \vdash \pi_1 \mathsf{M}_1 = \pi_1 \mathsf{M}_2 : \mathsf{A}_1 \quad \Gamma \vdash \pi_2 \mathsf{M}_1 = \pi_2 \mathsf{M}_2 : \mathsf{A}_2 \, [\pi_1 \mathsf{M}_1 / \mathsf{x}]\end{array}}{\Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2}$$

Symmetry

$$\frac{\Gamma \vdash \mathsf{M}_2 = \mathsf{M}_1 : \mathsf{A}}{\Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : \mathsf{A}}$$

Transitivity

$$\frac{\Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : \mathsf{A} \qquad \Gamma \vdash \mathsf{M}_2 = \mathsf{M}_3 : \mathsf{A}}{\Gamma \vdash \mathsf{M}_1 = \mathsf{M}_3 : \mathsf{A}}$$

Type Conversion

$$\frac{\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_2 : \mathsf{type} \qquad \Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : \mathsf{A}_1}{\Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : \mathsf{A}_2}$$

$\boxed{\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_2 : \mathsf{K}}$

Constants

$$\frac{\Sigma(\mathsf{a}) = \mathsf{K}}{\Gamma \vdash \mathsf{a} = \mathsf{a} : \mathsf{K}}$$

Abstractions

$$\frac{\Gamma \vdash \mathsf{A}_{11} = \mathsf{A}_1 : \mathsf{type} \quad \Gamma \vdash \mathsf{A}_{21} = \mathsf{A}_1 : \mathsf{type} \quad \Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{A}_{12} = \mathsf{A}_{22} : \mathsf{K}}{\Gamma \vdash \lambda \mathsf{x}{:}\mathsf{A}_{11}.\mathsf{A}_{12} = \lambda \mathsf{x}{:}\mathsf{A}_{21}.\mathsf{A}_{22} : \Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{K}}$$

Applications

$$\frac{\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_2 : \Pi \mathsf{x}{:}\mathsf{A}_3.\mathsf{K} \qquad \Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : \mathsf{A}_3}{\Gamma \vdash \mathsf{A}_1 \, \mathsf{M}_1 = \mathsf{A}_2 \, \mathsf{M}_2 : \mathsf{K} \, [\mathsf{M}_1 / \mathsf{x}]}$$

Extensionality
at Π kind

$$\frac{\Gamma \vdash \mathsf{A} : \mathsf{type} \quad \Gamma \vdash \mathsf{A}_1 : \Pi \mathsf{x}{:}\mathsf{A}.\mathsf{K} \quad \Gamma \vdash \mathsf{A}_2 : \Pi \mathsf{x}{:}\mathsf{A}.\mathsf{K} \quad \Gamma, \mathsf{x}{:}\mathsf{A} \vdash \mathsf{A}_1 \, \mathsf{x} = \mathsf{A}_2 \, \mathsf{x} : \mathsf{K}}{\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_2 : \Pi \mathsf{x}{:}\mathsf{A}.\mathsf{K}}$$

Parallel β-Conversion
for function application

$$\frac{\Gamma \vdash \mathsf{A} : \mathsf{type} \qquad \Gamma, \mathsf{x}{:}\mathsf{A} \vdash \mathsf{A}_1 = \mathsf{A}_2 : \mathsf{K} \qquad \Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : \mathsf{A}}{\Gamma \vdash (\lambda \mathsf{x}{:}\mathsf{A}.\mathsf{A}_1) \, \mathsf{M}_1 = \mathsf{A}_2 \, [\mathsf{M}_2 / \mathsf{x}] : \mathsf{K} \, [\mathsf{M}_1 / \mathsf{x}]}$$

Dependent Function Types

$$\frac{\Gamma \vdash \mathsf{A}_{11} = \mathsf{A}_{21} : \mathsf{type} \qquad \Gamma \vdash \mathsf{A}_{11} : \mathsf{type} \qquad \Gamma, \mathsf{x}{:}\mathsf{A}_{11} \vdash \mathsf{A}_{12} = \mathsf{A}_{22} : \mathsf{type}}{\Gamma \vdash \Pi \mathsf{x}{:}\mathsf{A}_{11}.\mathsf{A}_{12} = \Pi \mathsf{x}{:}\mathsf{A}_{21}.\mathsf{A}_{22} : \mathsf{type}}$$

Dependent Pair Types

$$\frac{\Gamma \vdash \mathsf{A}_{11} = \mathsf{A}_{21} : \mathsf{type} \qquad \Gamma \vdash \mathsf{A}_{11} : \mathsf{type} \qquad \Gamma, \mathsf{x}{:}\mathsf{A}_{11} \vdash \mathsf{A}_{12} = \mathsf{A}_{22} : \mathsf{type}}{\Gamma \vdash \Sigma \mathsf{x}{:}\mathsf{A}_{11}.\mathsf{A}_{12} = \Sigma \mathsf{x}{:}\mathsf{A}_{21}.\mathsf{A}_{22} : \mathsf{type}}$$

Unit

$$\frac{}{\Gamma \vdash 1 = 1 : \mathsf{type}}$$

Symmetry

$$\frac{\Gamma \vdash \mathsf{A}_2 = \mathsf{A}_1 : \mathsf{K}}{\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_2 : \mathsf{K}}$$

Transitivity

$$\frac{\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_2 : \mathsf{K} \qquad \Gamma \vdash \mathsf{A}_2 = \mathsf{A}_3 : \mathsf{K}}{\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_3 : \mathsf{K}}$$

Kind Conversion

$$\frac{\Gamma \vdash \mathsf{K}_1 = \mathsf{K}_2 : \mathsf{kind} \qquad \Gamma \vdash \mathsf{A}_1 = \mathsf{A}_2 : \mathsf{K}_1}{\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_2 : \mathsf{K}_2}$$

$\boxed{\Gamma \vdash \mathsf{K}_1 = \mathsf{K}_2 : \mathsf{kind}}$

Type

$$\frac{}{\Gamma \vdash \mathsf{type} = \mathsf{type} : \mathsf{kind}}$$

Dependent Function Kinds

$$\frac{\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_2 : \mathsf{type} \qquad \Gamma \vdash \mathsf{A}_1 : \mathsf{type} \qquad \Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{K}_1 = \mathsf{K}_2 : \mathsf{kind}}{\Gamma \vdash \Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{K}_1 = \Pi \mathsf{x}{:}\mathsf{A}_2.\mathsf{K}_2 : \mathsf{kind}}$$

Symmetry

$$\frac{\Gamma \vdash \mathsf{K}_2 = \mathsf{K}_1 : \mathsf{kind}}{\Gamma \vdash \mathsf{K}_1 = \mathsf{K}_2 : \mathsf{kind}}$$

Transitivity

$$\frac{\Gamma \vdash \mathsf{K}_1 = \mathsf{K}_2 : \mathsf{kind} \qquad \Gamma \vdash \mathsf{K}_2 = \mathsf{K}_3 : \mathsf{kind}}{\Gamma \vdash \mathsf{K}_1 = \mathsf{K}_3 : \mathsf{kind}}$$

Well Typed Substitutions Finally, we introduce notation for typing substitutions below.

Definition 3.1 *The judgment* $\Gamma_2 \vdash \sigma : \Gamma_1$ *holds iff* $\forall \mathsf{x} \in \mathsf{Dom}(\Gamma_1).\Gamma_2 \vdash \sigma(\mathsf{x}) : \sigma(\Gamma_1(\mathsf{x}))$.

Definition 3.2 *The judgment* $\Gamma_2 \vdash \sigma_1 = \sigma_2 : \Gamma_1$ *holds iff*

- $\Gamma_2 \vdash \sigma_1 : \Gamma_1$,
- $\Gamma_2 \vdash \sigma_2 : \Gamma_1$, *and*
- $\forall \mathsf{x} \in \mathsf{Dom}(\Gamma_1).\Gamma_2 \vdash \sigma_1(\mathsf{x}) = \sigma_2(\mathsf{x}) : \sigma_1(\Gamma_1(\mathsf{x}))$.

4 Elementary Properties

We start with some basic properties of the system. The most important lemma in this section is the regularity lemma, which states that terms appearing in valid derivations are themselves well typed. To get there, we first prove some basic structural facts about the typing judgments can be proved by easy structural inductions.

Lemma 4.1 (Weakening) *If* $\Gamma_1 \vdash \mathcal{J}$ *and* $\Gamma_1 \subseteq \Gamma_2$*, then* $\Gamma_2 \vdash \mathcal{J}$*.*

Proof

By structural induction on the derivation of the first judgment.

□

Lemma 4.2 (Free Variable Containment) *If* $\vdash \Gamma : \mathsf{ctx}$ *and* $\Gamma \vdash \mathcal{J}$*, then* $FV(\mathcal{J}) \in \mathsf{Dom}(\Gamma)$*.*

Proof

By structural induction on the derivation of the second judgment.

□

Next we show that reflexivity is admissible for our definition of definitional equality. This lemma shows that definitional equality is an equivalence relation (it is already symmetric and transitive by rules).

Lemma 4.3 (Reflexivity)

1. *If* $\Gamma \vdash \mathsf{M} : \mathsf{A}$ *then* $\Gamma \vdash \mathsf{M} = \mathsf{M} : \mathsf{A}$*.*
2. *If* $\Gamma \vdash \mathsf{A} : \mathsf{K}$ *then* $\Gamma \vdash \mathsf{A} = \mathsf{A} : \mathsf{K}$*.*
3. *If* $\Gamma \vdash \mathsf{K} : \mathsf{kind}$ *then* $\Gamma \vdash \mathsf{K} = \mathsf{K} : \mathsf{kind}$*.*

Proof

By structural induction on the derivation of the judgment.

□

One of the key properties of a declarative system is the admissibility of a substitution property for hypotheses. To prove this, we need some basic facts about substitutions, such as that we can always come up with an identity substitution for a well-typed context, and that substitutions can be extended to have no effect on terms not in the domain of the substitution.

Lemma 4.4 (Identity Substitutions) *If* $\vdash \Gamma : \mathsf{ctx}$ *then* $\Gamma \vdash \mathsf{id}_\Gamma : \Gamma$ *and* $\Gamma \vdash \mathsf{id}_\Gamma = \mathsf{id}_\Gamma : \Gamma$*.*

Proof

By induction on the construction of the context.

□

Lemma 4.5 (Extending Substitutions)

1. *If* $\Gamma_1 \vdash \sigma_1 : \Gamma$*,* $\Gamma \vdash \mathsf{A} : \mathsf{type}$ *and* $\mathsf{x} \notin \mathsf{Dom}(\Gamma) \cup \mathsf{Dom}(\Gamma_1)$ *then* $\Gamma_1, \mathsf{x}{:}\mathsf{A}[\sigma_1] \vdash \sigma_1, \mathsf{x}/\mathsf{x} : \Gamma, \mathsf{x}{:}\mathsf{A}$*.*
2. *If* $\Gamma_1 \vdash \sigma_1 = \sigma_2 : \Gamma$*,* $\Gamma \vdash \mathsf{A} : \mathsf{type}$ *and* $\mathsf{x} \notin \mathsf{Dom}(\Gamma) \cup \mathsf{Dom}(\Gamma_1)$ *then* $\Gamma_1, \mathsf{x}{:}\mathsf{A}[\sigma_1] \vdash \sigma_1, \mathsf{x}/\mathsf{x} = \sigma_2, \mathsf{x}/\mathsf{x} : \Gamma, \mathsf{x}{:}\mathsf{A}$*.*

Proof

Case 1:

Direct, by the definition of substitution typing, using Weakening.

Case 2:

Direct, by the definition of substitution equality, part 1 and Weakening. □

Next, we show that the notion of substitutions can be lifted to entire judgments.

Lemma 4.6 (Substitution) *If* $\Gamma_1 \vdash \mathcal{J}$ *and* $\Gamma_2 \vdash \sigma : \Gamma_1$*, then* $\Gamma_2 \vdash \mathcal{J}[\sigma]$*.*

Proof

By structural induction on the second judgment.

□

With the key substitution property already proved, an useful fact about contexts can now be proved. We often want to change declarations in the context to bind a definitionally equal type. This is now easy to prove.

Lemma 4.7 (Context Conversion) *Assume* $\vdash \Gamma, \mathsf{x}{:}\mathsf{A}_1 : \mathsf{ctx}$*,* $\Gamma \vdash \mathsf{A}_2 : \mathsf{type}$*, and* $\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_2 : \mathsf{type}$*. If* $\Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathcal{J}$*, then* $\Gamma, \mathsf{x}{:}\mathsf{A}_2 \vdash \mathcal{J}$*.*

Proof

Direct, by weakening and substitution.

$\Gamma, \mathsf{x}{:}\mathsf{A}_2 \vdash \mathsf{x} : \mathsf{A}_2$	By rule (variable)
$\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_2 : \mathsf{type}$	By assumption
$\Gamma \vdash \mathsf{A}_2 = \mathsf{A}_1 : \mathsf{type}$	By rule (symmetry)
$\Gamma, \mathsf{x}{:}\mathsf{A}_2 \vdash \mathsf{x} : \mathsf{A}_1$	By rule (type conversion)
$\Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathcal{J}$	By assumption
$\Gamma, \mathsf{x}_1{:}\mathsf{A}_1 \vdash \mathcal{J}[\mathsf{x}_1/\mathsf{x}]$	By substitution
$\Gamma, \mathsf{x}{:}\mathsf{A}_2, \mathsf{x}_1{:}\mathsf{A}_1 \vdash \mathcal{J}[\mathsf{x}_1/\mathsf{x}]$	By weakening
$\Gamma, \mathsf{x}{:}\mathsf{A}_2 \vdash (\mathcal{J}[\mathsf{x}_1/\mathsf{x}])[\mathsf{x}/\mathsf{x}_1]$	By substitution
$\Gamma, \mathsf{x}{:}\mathsf{A}_2 \vdash \mathcal{J}$	By definition of substitution

□

A critical lemma for our purposes is the lemma that is usually called functionality. This says that equal substitutions are "functional", *i.e.* given equal elements, they produce equal elements. Such a property is needed, for example, in our proofs of completeness of algorithmic equality with respect to the definitional equality given above. Our proof of this property needs regularity. However, our proof of regularity itself seems to need this property. Fortunately, the form of functionality needed in the regularity proof is less general. We need the fact that applying equal substitutions to *the same* element produces equal elements. This less general lemma can be proved directly, without an appeal to regularity. After we prove regularity, we can come back and prove the more general version we will need later.

Lemma 4.8 (Functionality for Typing) *Assume* $\Gamma_1 \vdash \sigma_1 : \Gamma$*,* $\Gamma_1 \vdash \sigma_2 : \Gamma$*, and* $\Gamma_1 \vdash \sigma_1 = \sigma_2 : \Gamma$*.*

1. *If* $\Gamma \vdash \mathsf{M} : \mathsf{A}$ *then* $\Gamma_1 \vdash \mathsf{M}[\sigma_1] = \mathsf{M}[\sigma_2] : \mathsf{A}[\sigma_1]$*.*
2. *If* $\Gamma \vdash \mathsf{A} : \mathsf{K}$ *then* $\Gamma_1 \vdash \mathsf{A}[\sigma_1] = \mathsf{A}[\sigma_2] : \mathsf{K}[\sigma_1]$*.*
3. *If* $\Gamma \vdash \mathsf{K} : \mathsf{kind}$ *then* $\Gamma_1 \vdash \mathsf{K}[\sigma_1] = \mathsf{K}[\sigma_2] : \mathsf{kind}$*.*

Proof

By induction on the typing judgment. We show one representative cases for pairs.

Case 1:
$$\frac{\Gamma \vdash \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 : \mathsf{type} \qquad \Gamma \vdash \mathsf{M}_1 : \mathsf{A}_1 \qquad \Gamma \vdash \mathsf{M}_2 : \mathsf{A}_2\,[\mathsf{M}_1/\mathsf{x}]}{\Gamma \vdash \langle \mathsf{M}_1, \mathsf{M}_2 \rangle^{\Sigma \mathsf{x}:\mathsf{A}_1.\mathsf{A}_2} : \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2}$$

$\Gamma_1 \vdash (\Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2)[\sigma_1] : \mathsf{type}$	By Substitution
$\Gamma_1 \vdash \Sigma \mathsf{x}{:}(\mathsf{A}_1[\sigma_1]).(\mathsf{A}_2[\sigma_1]) : \mathsf{type}$	By definition of substitution
$\Gamma_1 \vdash \mathsf{M}_1[\sigma_1] = \mathsf{M}_1[\sigma_2] : \mathsf{A}_1[\sigma_1]$	By induction
$\Gamma_1 \vdash \mathsf{M}_2[\sigma_1] = \mathsf{M}_1[\sigma_2] : (\mathsf{A}_2\,[\mathsf{M}_1/\mathsf{x}])[\sigma_1]$	By induction

$\Gamma_1 \vdash \mathsf{M}_2[\sigma_1] = \mathsf{M}_1[\sigma_2] : (\mathsf{A}_2[\sigma_1])\,[(\mathsf{M}_1[\sigma_1])/\mathsf{x}]$ By properties of substitution
$\Gamma_1 \vdash \langle \mathsf{M}_1[\sigma_1], \mathsf{M}_2[\sigma_1]\rangle^{\Sigma \mathsf{x}:(\mathsf{A}_1[\sigma_1]).(\mathsf{A}_2[\sigma_1])} = \langle \mathsf{M}_1[\sigma_2], \mathsf{M}_2[\sigma_2]\rangle^{\Sigma \mathsf{x}:(\mathsf{A}_1[\sigma_2]).(\mathsf{A}_2[\sigma_2])} : \Sigma \mathsf{x}{:}(\mathsf{A}_1[\sigma_1]).(\mathsf{A}_2[\sigma_1])$
By rule (products)
$\Gamma_1 \vdash (\langle \mathsf{M}_1, \mathsf{M}_2\rangle^{\Sigma \mathsf{x}:\mathsf{A}_1.\mathsf{A}_2})[\sigma_1] = (\langle \mathsf{M}_1, \mathsf{M}_2\rangle^{\Sigma \mathsf{x}:\mathsf{A}_1.\mathsf{A}_2})[\sigma_2] : (\Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2)[\sigma_1$ By definition of substitution

□

One last lemma is needed before we give our proof of regularity. We need for sum and product types and product kinds the fact that the components of these types are well-formed if the types themselves are well-formed. This is applied in the sum and product type cases in the proof of regularity.

Lemma 4.9 (Inversion on Products and Sums)

1. *If* $\Gamma \vdash \Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 : \mathsf{K}$ *then* $\Gamma \vdash \mathsf{A}_1 : \mathsf{type}$ *and* $\Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{A}_2 : \mathsf{type}$.
2. *If* $\Gamma \vdash \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 : \mathsf{K}$ *then* $\Gamma \vdash \mathsf{A}_1 : \mathsf{type}$ *and* $\Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{A}_2 : \mathsf{type}$.
3. *If* $\Gamma \vdash \Pi \mathsf{x}{:}\mathsf{A}.\mathsf{K} : \mathsf{kind}$ *then* $\Gamma \vdash \mathsf{A} : \mathsf{type}$ *and* $\Gamma, \mathsf{x}{:}\mathsf{A} \vdash \mathsf{K} : \mathsf{kind}$.

Proof

By induction on the first derivation.

□

We are now in a position to prove our key regularity property.

Lemma 4.10 (Regularity) *Assume* $\vdash \Gamma : \mathsf{ctx}$.

1. *If* $\Gamma \vdash \mathsf{M} : \mathsf{A}$ *then* $\Gamma \vdash \mathsf{A} : \mathsf{type}$.
2. *If* $\Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : \mathsf{A}$ *then* $\Gamma \vdash \mathsf{M}_1 : \mathsf{A}$, $\Gamma \vdash \mathsf{M}_2 : \mathsf{A}$, *and* $\Gamma \vdash \mathsf{A} : \mathsf{type}$.
3. *If* $\Gamma \vdash \mathsf{A} : \mathsf{K}$ *then* $\Gamma \vdash \mathsf{K} : \mathsf{kind}$.
4. *If* $\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_2 : \mathsf{K}$ *then* $\Gamma \vdash \mathsf{A}_1 : \mathsf{K}$, $\Gamma \vdash \mathsf{A}_2 : \mathsf{K}$, *and* $\Gamma \vdash \mathsf{K} : \mathsf{kind}$.
5. *If* $\Gamma \vdash \mathsf{K}_1 = \mathsf{K}_2 : \mathsf{kind}$ *then* $\Gamma \vdash \mathsf{K}_1 : \mathsf{kind}$, *and* $\Gamma \vdash \mathsf{K}_2 : \mathsf{kind}$.

Proof

By induction on the derivation of the judgment. We show some representative cases.

Case 1: $$\frac{\Gamma \vdash \mathsf{A}_1 : \mathsf{type} \qquad \Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{M}_2 : \mathsf{A}_2}{\Gamma \vdash \lambda \mathsf{x}{:}\mathsf{A}_1.\mathsf{M}_2 : \Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2}$$

$\Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{A}_2 : \mathsf{type}$ By induction
$\Gamma \vdash \Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 : \mathsf{type}$ By rule

Case 2: $$\frac{\Gamma \vdash \mathsf{M} : \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2}{\Gamma \vdash \pi_2 \mathsf{M} : \mathsf{A}_2\,[\pi_1 \mathsf{M}/\mathsf{x}]}$$

$\Gamma \vdash \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 : \mathsf{type}$ By induction
$\Gamma \vdash \mathsf{A}_1 : \mathsf{type}$,
$\Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{A}_2 : \mathsf{type}$ By Inversion on Sums
$\Gamma \vdash \mathsf{M} : \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2$ By assumption
$\Gamma \vdash \pi_1 \mathsf{M} : \mathsf{A}_1$ By rule (projection)
$\Gamma_2 \vdash \mathsf{A}_2\,[\pi_1 \mathsf{M}/\mathsf{x}] : \mathsf{type}$ By Substitution

Case 3: $$\frac{\Gamma \vdash \mathsf{A}_{11} = \mathsf{A}_1 : \mathsf{type} \qquad \Gamma \vdash \mathsf{A}_{12} = \mathsf{A}_1 : \mathsf{type} \qquad \Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{M}_1 = \mathsf{M}_2 : \mathsf{A}_2}{\Gamma \vdash \lambda \mathsf{x}{:}\mathsf{A}_{11}.\mathsf{M}_1 = \lambda \mathsf{x}{:}\mathsf{A}_{12}.\mathsf{M}_2 : \Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2}$$

$\Gamma \vdash \mathsf{A}_{11} : \mathsf{type}$	By induction
$\Gamma \vdash \mathsf{A}_{21} : \mathsf{type}$	By induction
$\Gamma \vdash \mathsf{A}_1 : \mathsf{type}$	By induction
$\Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{M}_1 : \mathsf{A}_2,$	
$\Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{A}_2 : \mathsf{A}_2,$	
$\Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{A}_2 : \mathsf{type}$	By induction
$\Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{A}_2 = \mathsf{A}_2 : \mathsf{type}$	By Reflexivity
$\Gamma \vdash \Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 : \mathsf{type}$	By rule
$\Gamma \vdash \Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 = \Pi \mathsf{x}{:}\mathsf{A}_{11}.\mathsf{A}_2 : \mathsf{type}$	By rule
$\Gamma \vdash \Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 = \Pi \mathsf{x}{:}\mathsf{A}_{21}.\mathsf{A}_2 : \mathsf{type}$	By rule
$\Gamma, \mathsf{x}{:}\mathsf{A}_{11} \vdash \mathsf{M}_1 : \mathsf{A}_2$	By Context Conversion
$\Gamma \vdash \lambda \mathsf{x}{:}\mathsf{A}_{11}.\mathsf{M}_1 : \Pi \mathsf{x}{:}\mathsf{A}_{11}.\mathsf{A}_2$	By rule (abstraction)
$\Gamma_1 \vdash \lambda \mathsf{x}{:}\mathsf{A}_{11}.\mathsf{M}_1 : \Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2$	By rule (type conversion)
$\Gamma, \mathsf{x}{:}\mathsf{A}_{21} \vdash \mathsf{M}_2 : \mathsf{A}_2$	By Context Conversion
$\Gamma_1 \vdash \lambda \mathsf{x}{:}\mathsf{A}_{21}.\mathsf{M}_2 : \Pi \mathsf{x}{:}\mathsf{A}_{21}.\mathsf{A}_2$	By rule (abstraction)
$\Gamma_1 \vdash \lambda \mathsf{x}{:}\mathsf{A}_{21}.\mathsf{M}_2 : \Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2$	By rule (type conversion)

Case 4: $$\frac{\Gamma \vdash \mathsf{A}_1 : \mathsf{type} \qquad \Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{M}_{12} = \mathsf{M}_{22} : \mathsf{A}_2 \qquad \Gamma \vdash \mathsf{M}_{11} = \mathsf{M}_{21} : \mathsf{A}_1}{\Gamma \vdash (\lambda \mathsf{x}{:}\mathsf{A}_1.\mathsf{M}_{12})\, \mathsf{M}_{11} = \mathsf{M}_{22}\,[\mathsf{M}_{21}/\mathsf{x}] : \mathsf{A}_2\,[\mathsf{M}_{11}/\mathsf{x}]}$$

$\Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{M}_{12} : \mathsf{A}_2,$	
$\Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{M}_{22} : \mathsf{A}_2,$	
$\Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{A}_2 : \mathsf{type}$	By induction
$\Gamma \vdash \mathsf{M}_{11} : \mathsf{A}_1,$	
$\Gamma \vdash \mathsf{M}_{21} : \mathsf{A}_1,$	
$\Gamma \vdash \mathsf{A}_1 : \mathsf{type}$	By induction
$\Gamma \vdash \mathsf{A}_2\,[\mathsf{M}_{11}/\mathsf{x}] : \mathsf{type}$	By Substitution
$\Gamma \vdash \mathsf{A}_2\,[\mathsf{M}_{11}/\mathsf{x}] = \mathsf{A}_2\,[\mathsf{M}_{21}/\mathsf{x}] : \mathsf{type}$	By Functionality
$\Gamma \vdash \mathsf{M}_{22}\,[\mathsf{M}_{21}/\mathsf{x}] : \mathsf{A}_2\,[\mathsf{M}_{21}/\mathsf{x}]$	By Substitution
$\Gamma \vdash \mathsf{M}_{22}\,[\mathsf{M}_{21}/\mathsf{x}] : \mathsf{A}_2\,[\mathsf{M}_{11}/\mathsf{x}]$	By rule (type conversion)
$\Gamma \vdash \lambda \mathsf{x}{:}\mathsf{A}_1.\mathsf{M}_{12} : \Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2$	By rule
$\Gamma \vdash (\lambda \mathsf{x}{:}\mathsf{A}_1.\mathsf{M}_{12})\, \mathsf{M}_{11} : \mathsf{A}_2\,[\mathsf{M}_{11}/\mathsf{x}]$	By rule

Case 5: $$\frac{\Gamma \vdash \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 : \mathsf{type} \qquad \Gamma \vdash \mathsf{M}_{11} = \mathsf{M}_{21} : \mathsf{A}_1 \qquad \Gamma \vdash \mathsf{M}_{12} = \mathsf{M}_{22} : \mathsf{A}_2\,[\mathsf{M}_{11}/\mathsf{x}]}{\Gamma \vdash \langle \mathsf{M}_{11}, \mathsf{M}_{12} \rangle^{\Sigma \mathsf{x}:\mathsf{A}_1.\mathsf{A}_2} = \langle \mathsf{M}_{21}, \mathsf{M}_{22} \rangle^{\Sigma \mathsf{x}:\mathsf{A}_1.\mathsf{A}_2} : \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2}$$

$\Gamma \vdash \mathsf{A}_1 : \mathsf{type},$	
$\Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{A}_2 : \mathsf{type}$	By Inversion on Sums
$\Gamma \vdash \mathsf{M}_{11} : \mathsf{A}_1,$	
$\Gamma \vdash \mathsf{M}_{21} : \mathsf{A}_1$	By induction
$\Gamma \vdash \mathsf{A}_2\,[\mathsf{M}_{11}/\mathsf{x}] = \mathsf{A}_2\,[\mathsf{M}_{21}/\mathsf{x}] : \mathsf{type}$	By Functionality
$\Gamma \vdash \mathsf{M}_{12} : \mathsf{A}_2\,[\mathsf{M}_{11}/\mathsf{x}],$	
$\Gamma \vdash \mathsf{M}_{22} : \mathsf{A}_2\,[\mathsf{M}_{11}/\mathsf{x}]$	By induction
$\Gamma \vdash \mathsf{M}_{22} : \mathsf{A}_2\,[\mathsf{M}_{21}/\mathsf{x}]$	By rule (type conversion)
$\Gamma \vdash \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 : \mathsf{type}$	By assumption
$\Gamma \vdash \langle \mathsf{M}_{11}, \mathsf{M}_{12} \rangle^{\Sigma \mathsf{x}:\mathsf{A}_1.\mathsf{A}_2} : \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2$	By rule
$\Gamma \vdash \langle \mathsf{M}_{21}, \mathsf{M}_{22} \rangle^{\Sigma \mathsf{x}:\mathsf{A}_1.\mathsf{A}_2} : \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2$	By rule

Case 6: $$\frac{\Gamma \vdash \mathsf{M}_1 : \mathsf{A}_1 \qquad \Gamma \vdash \mathsf{M}_2 = \mathsf{M}_3 : \mathsf{A}_2}{\Gamma \vdash \pi_2 \langle \mathsf{M}_1, \mathsf{M}_2 \rangle^{\mathsf{A}} = \mathsf{M}_3 : \mathsf{A}_2}$$

$\Gamma \vdash \mathsf{M}_2 : \mathsf{A}_2$,	
$\Gamma \vdash \mathsf{M}_3 : \mathsf{A}_2$,	
$\Gamma \vdash \mathsf{A}_2 : \mathsf{type}$	By induction
$\Gamma \vdash \mathsf{A}_1 : \mathsf{type}$	By induction
$\Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{A}_2 : \mathsf{type}$	By Weakening
$\Gamma \vdash \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 : \mathsf{type}$	By rule
$\Gamma \vdash \mathsf{M}_2 : \mathsf{A}_2\,[\mathsf{M}_1/\mathsf{x}]$	Since $\mathsf{x} \notin FV(\mathsf{A}_2)$
$\Gamma \vdash \langle \mathsf{M}_1, \mathsf{M}_2 \rangle^{\mathsf{A}} : \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2$	By rule (pair)
$\Gamma \vdash \pi_2 \langle \mathsf{M}_1, \mathsf{M}_2 \rangle^{\mathsf{A}} : \mathsf{A}_2\,[\pi_1 \mathsf{M}_1/\mathsf{x}]$	By rule (projection)
$\Gamma \vdash \pi_2 \langle \mathsf{M}_1, \mathsf{M}_2 \rangle^{\mathsf{A}} : \mathsf{A}_2$	Since $\mathsf{x} \notin FV(\mathsf{A}_2)$

□

With regularity now proved, we can prove two important lemmas used many times in the later sections. One is a generalization of the functionality property to the equality judgments. The other is an inversion on typing judgments.

Lemma 4.11 (Functionality for Equality) *Assume* $\Gamma_1 \vdash \sigma_1 = \sigma_2 : \Gamma$.

1. *If* $\Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : \mathsf{A}$ *then* $\Gamma_1 \vdash \mathsf{M}_1[\sigma_1] = \mathsf{M}_2[\sigma_2] : \mathsf{A}[\sigma_1]$.
2. *If* $\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_2 : \mathsf{K}$ *then* $\Gamma_1 \vdash \mathsf{A}_1[\sigma_1] = \mathsf{A}_2[\sigma_2] : \mathsf{K}[\sigma_1]$.
3. *If* $\Gamma \vdash \mathsf{K}_1 = \mathsf{K}_2 : \mathsf{kind}$ *then* $\Gamma_1 \vdash \mathsf{K}_1[\sigma_1] = \mathsf{K}_2[\sigma_2] : \mathsf{kind}$.

Proof

Direct. We show one case.

$\Gamma_1 \vdash \sigma_1 : \Gamma$,	
$\Gamma_1 \vdash \sigma_2 : \Gamma$	By definition of substitution equality
$\Gamma_1 \vdash \mathsf{M}_1[\sigma_1] = \mathsf{M}_2[\sigma_1] : \mathsf{A}[\sigma_1]$	By Substitution
$\Gamma \vdash \mathsf{M}_2 : \mathsf{A}$	By Regularity
$\Gamma_1 \vdash \mathsf{M}_2[\sigma_1] = \mathsf{M}_2[\sigma_2] : \mathsf{A}[\sigma_1]$	By Functionality
$\Gamma_1 \vdash \mathsf{M}_1[\sigma_1] = \mathsf{M}_2[\sigma_2] : \mathsf{A}[\sigma_1]$	By rule (transitivity)

□

Lemma 4.12 (Typing Inversion) *Assume* $\vdash \Gamma : \mathsf{ctx}$.

1. *If* $\Gamma \vdash \mathsf{x} : \mathsf{A}_1$ *then* $\Gamma(\mathsf{x}) = \mathsf{A}_2$ *and* $\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_2 : \mathsf{type}$.
2. *If* $\Gamma \vdash \mathsf{c} : \mathsf{A}_1$ *then* $\Sigma(\mathsf{c}) = \mathsf{A}_2$ *and* $\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_2 : \mathsf{type}$.
3. *If* $\Gamma \vdash \lambda \mathsf{x}{:}\mathsf{A}_1.\mathsf{M}_1 : \mathsf{A}$ *then* $\Gamma \vdash \mathsf{A}_1 : \mathsf{type}$ *and* $\Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{M}_1 : \mathsf{A}_2$ *and* $\Gamma \vdash \Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 = \mathsf{A} : \mathsf{type}$.
4. *If* $\Gamma \vdash \mathsf{M}_1\,\mathsf{M}_2 : \mathsf{A}$ *then* $\Gamma \vdash \mathsf{M}_1 : \Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2$, $\Gamma \vdash \mathsf{M}_2 : \mathsf{A}_1$ *and* $\Gamma \vdash \mathsf{A}_2\,[\mathsf{M}_2/\mathsf{x}] = \mathsf{A} : \mathsf{type}$.
5. *If* $\Gamma \vdash \langle \mathsf{M}_1, \mathsf{M}_2 \rangle^{\Sigma \mathsf{x}:\mathsf{A}_1.\mathsf{A}_2} : \mathsf{A}$ *then* $\Gamma \vdash \mathsf{M}_1 : \mathsf{A}_1$, $\Gamma \vdash \mathsf{M}_2 : \mathsf{A}_2\,[\mathsf{M}_1/\mathsf{x}]$ *and* $\Gamma \vdash \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 = \mathsf{A} : \mathsf{type}$.
6. *If* $\Gamma \vdash \pi_1 \mathsf{M}_1 : \mathsf{A}$ *then* $\Gamma \vdash \mathsf{M}_1 : \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2$ *and* $\Gamma \vdash \mathsf{A}_1 = \mathsf{A} : \mathsf{type}$.

7. *If* $\Gamma \vdash \pi_2 \mathsf{M}_1 : \mathsf{A}$ *then* $\Gamma \vdash \mathsf{M}_1 : \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2$ *and* $\Gamma \vdash \mathsf{A}_2\,[\pi_1 \mathsf{M}_1/\mathsf{x}] = \mathsf{A} : \mathsf{type}$.

8. *If* $\Gamma \vdash \langle\rangle : \mathsf{A}$ *then* $\Gamma \vdash 1 = \mathsf{A} : \mathsf{type}$.

9. *If* $\Gamma \vdash \mathsf{a} : \mathsf{K}_1$ *then* $\Sigma(\mathsf{a}) = \mathsf{K}_2$ *and* $\Gamma \vdash \mathsf{K}_1 = \mathsf{K}_2 : \mathsf{kind}$.

10. *If* $\Gamma \vdash \lambda \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 : \mathsf{K}$ *then* $\Gamma \vdash \mathsf{A}_1 : \mathsf{type}$ *and* $\Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{A}_2 : \mathsf{K}_1$ *and* $\Gamma \vdash \Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{K}_1 = \mathsf{K} : \mathsf{kind}$.

11. *If* $\Gamma \vdash \mathsf{A}\,\mathsf{M} : \mathsf{K}$ *then* $\Gamma \vdash \mathsf{A} : \Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{K}_1$, $\Gamma \vdash \mathsf{M} : \mathsf{A}_1$ *and* $\Gamma \vdash \mathsf{K}_1\,[\mathsf{M}/\mathsf{x}] = \mathsf{K} : \mathsf{kind}$.

12. *If* $\Gamma \vdash \Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 : \mathsf{K}$ *then* $\Gamma \vdash \mathsf{A}_1 : \mathsf{type}$, $\Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{A}_2 : \mathsf{type}$ *and* $\Gamma \vdash \mathsf{type} = \mathsf{K} : \mathsf{kind}$.

13. *If* $\Gamma \vdash \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 : \mathsf{K}$ *then* $\Gamma \vdash \mathsf{A}_1 : \mathsf{type}$, $\Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{A}_2 : \mathsf{type}$ *and* $\Gamma \vdash \mathsf{type} = \mathsf{K} : \mathsf{kind}$.

14. *If* $\Gamma \vdash 1 : \mathsf{K}$ *then* $\Gamma \vdash \mathsf{type} = \mathsf{K} : \mathsf{kind}$.

Proof

By induction on the derivation of the judgment. We show a few representative cases.

Case 1: $$\frac{\Gamma \vdash \mathsf{M} : \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2}{\Gamma \vdash \pi_2 \mathsf{M} : \mathsf{A}_2\,[\pi_1 \mathsf{M}/\mathsf{x}]}$$

$\Gamma \vdash \mathsf{A}_2\,[\pi_1 \mathsf{M}/\mathsf{x}] : \mathsf{type}$ By Regularity
$\Gamma \vdash \mathsf{A}_2\,[\pi_1 \mathsf{M}/\mathsf{x}] = \mathsf{A}_2\,[\pi_1 \mathsf{M}/\mathsf{x}] : \mathsf{AAtype}$ By Reflexivity

Case 2: $$\frac{\Gamma \vdash \langle \mathsf{M}_1, \mathsf{M}_2\rangle^{\mathsf{A}} : \mathsf{A}_1 \qquad \Gamma \vdash \mathsf{A}_1 = \mathsf{A}_2 : \mathsf{type}}{\Gamma \vdash \langle \mathsf{M}_1, \mathsf{M}_2\rangle^{\mathsf{A}} : \mathsf{A}_2}$$

$\Gamma \vdash \mathsf{M}_1 : \mathsf{A}_3$,
$\Gamma \vdash \mathsf{M}_2 : \mathsf{A}_4\,[\mathsf{M}_1/\mathsf{x}]$,
$\Gamma \vdash \Sigma \mathsf{x}{:}\mathsf{A}_3.\mathsf{A}_4 = \mathsf{A}_1 : \mathsf{type}$ By induction
$\Gamma \vdash \Sigma \mathsf{x}{:}\mathsf{A}_3.\mathsf{A}_4 = \mathsf{A}_2 : \mathsf{type}$ By rule (transitivity)

□

Lemma 4.13 (Extending Equal Substitutions) *If* $\vdash \Gamma : \mathsf{ctx}$, $\Gamma_1 \vdash \sigma_1 = \sigma_2 : \Gamma$, $\Gamma \vdash \mathsf{A} : \mathsf{type}$ *and* $\Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : \mathsf{A}$ *and* $\mathsf{x} \notin \mathsf{Dom}(\Gamma)$ *then* $\Gamma_1 \vdash \sigma_1, \mathsf{M}_1/\mathsf{x} = \sigma_2, \mathsf{M}_2/\mathsf{x} : \Gamma, \mathsf{x}{:}\mathsf{A}$.

Proof

Direct, by Regularity and Functionality for Equality. □

Lemma 4.14 (Equality Inversion on Kinds)

1. *If* $\Gamma \vdash \mathsf{K} = \mathsf{type} : \mathsf{kind}$ *or* $\Gamma \vdash \mathsf{type} = \mathsf{K} : \mathsf{kind}$ *then* $\mathsf{K} = \mathsf{type}$.

2. *If* $\Gamma \vdash \mathsf{K}' = \Pi \mathsf{x}{:}\mathsf{A}.\mathsf{K} : \mathsf{kind}$ *or* $\Gamma \vdash \Pi \mathsf{x}{:}\mathsf{A}.\mathsf{K} = \mathsf{K}' : \mathsf{kind}$ *then* $\mathsf{K}' = \Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{K}_1$ *with* $\Gamma \vdash \mathsf{A}_1 = \mathsf{A} : \mathsf{type}$ *and* $\Gamma \vdash \mathsf{K}_1 = \mathsf{K} : \mathsf{kind}$.

Proof

By induction on derivations.

□

5 Injectivity

We need a property usually called injectivity. Briefly, this says that given equal dependent function types, the components of the product types are themselves equal (at the kind type). A similar property should hold for dependent pair types. In Harper and Pfenning [5], this follows by an easy induction analogous to the case for product kinds (Lemma 4.14). However, we now have a non-trivial equality relation at the level of types due to the presence of family-level abstractions and applications. The problem in an inductive proof comes with the use of the transitivity rule. While judging equality between two products (say), the mediating family need not be of the same form. To get around this, we follow Vanderwaart and Crary [7] and prove by the method of logical relations. In defining the logical relation we will need, we need the auxiliary notion of weak-head reduction. This notion is important in its own right, and will be reused in the definition of the equality algorithm.

5.1 Weak Head Reduction

$\boxed{\mathsf{M}_1 \xrightarrow{\text{whr}} \mathsf{M}_2}$

$$\frac{}{(\lambda \mathsf{x}{:}\mathsf{A}_1.\mathsf{M}_2)\,\mathsf{M}_1 \xrightarrow{\text{whr}} \mathsf{M}_2\,[\mathsf{M}_1/\mathsf{x}]} \qquad \frac{\mathsf{M}_1 \xrightarrow{\text{whr}} \mathsf{M}_2}{\mathsf{M}_1\,\mathsf{M} \xrightarrow{\text{whr}} \mathsf{M}_2\,\mathsf{M}} \qquad \frac{}{\pi_i \langle \mathsf{M}_1, \mathsf{M}_2 \rangle^{\mathsf{A}} \xrightarrow{\text{whr}} \mathsf{M}_i} \qquad \frac{\mathsf{M}_1 \xrightarrow{\text{whr}} \mathsf{M}_2}{\pi_i \mathsf{M}_1 \xrightarrow{\text{whr}} \pi_i \mathsf{M}_2}$$

$\boxed{\mathsf{A}_1 \xrightarrow{\text{whr}} \mathsf{A}_2}$

$$\frac{}{(\lambda \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2)\,\mathsf{M} \xrightarrow{\text{whr}} \mathsf{A}_2\,[\mathsf{M}/\mathsf{x}]} \qquad \frac{\mathsf{A}_1 \xrightarrow{\text{whr}} \mathsf{A}_2}{\mathsf{A}_1\,\mathsf{M} \xrightarrow{\text{whr}} \mathsf{A}_2\,\mathsf{M}}$$

Lemma 5.1 (Determinacy of Weak Head Reduction)

1. *If* $\mathsf{M}_1 \xrightarrow{\text{whr}} \mathsf{M}_2$ *and* $\mathsf{M}_1 \xrightarrow{\text{whr}} \mathsf{M}_3$ *then* $\mathsf{M}_2 = \mathsf{M}_3$.
2. *If* $\mathsf{A}_1 \xrightarrow{\text{whr}} \mathsf{A}_2$ *and* $\mathsf{A}_1 \xrightarrow{\text{whr}} \mathsf{A}_3$ *then* $\mathsf{A}_2 = \mathsf{A}_3$.

Proof

By case analysis on the derivation of the first judgment.

□

We write $\xrightarrow{\text{whr}}^*$ for the reflexive transitive closure of the weak-head reduction relation $\xrightarrow{\text{whr}}$. On occasion, we write $\mathsf{M} \not\xrightarrow{\text{whr}}$ to indicate that there exists no M_1 such that $\mathsf{M} \xrightarrow{\text{whr}} \mathsf{M}_1$ according to the rules set out here. An analogous notation $\mathsf{A} \not\xrightarrow{\text{whr}}$ is defined for families.

5.2 A Logical Relation

The logical relation we use is defined by induction on kinds. The notable feature is that at higher kinds, we require definitional equality instead of logical relatedness for the arguments. Since the structure of kinds (*i.e.* whether they are a dependent function kind or type) is not affected by substitution of object variables, the definition of the relation is well-founded.

- $\Gamma \vdash \mathsf{A}_1 \approx \mathsf{A}_2 : [\![\mathsf{type}]\!]$ iff all these hold:
  - $\mathsf{A}_1 \xrightarrow{\text{whr}}^* \Pi \mathsf{x}{:}\mathsf{A}_{11}.\mathsf{A}_{12}$ iff $\mathsf{A}_2 \xrightarrow{\text{whr}}^* \Pi \mathsf{x}{:}\mathsf{A}_{21}.\mathsf{A}_{22}$.
  - $\mathsf{A}_1 \xrightarrow{\text{whr}}^* \Sigma \mathsf{x}{:}\mathsf{A}_{11}.\mathsf{A}_{12}$ iff $\mathsf{A}_2 \xrightarrow{\text{whr}}^* \Sigma \mathsf{x}{:}\mathsf{A}_{21}.\mathsf{A}_{22}$.

- If $\mathsf{A}_1 \stackrel{\text{whr}}{\longrightarrow}^* \Pi \mathsf{x}{:}\mathsf{A}_{11}.\mathsf{A}_{12}$ and $\mathsf{A}_2 \stackrel{\text{whr}}{\longrightarrow}^* \Pi \mathsf{x}{:}\mathsf{A}_{21}.\mathsf{A}_{22}$ then $\Gamma \vdash \mathsf{A}_{11} = \mathsf{A}_{21} : \mathsf{type}$ and $\Gamma, \mathsf{x}{:}\mathsf{A}_{11} \vdash \mathsf{A}_{12} = \mathsf{A}_{22} : \mathsf{type}$.
- If $\mathsf{A}_1 \stackrel{\text{whr}}{\longrightarrow}^* \Sigma \mathsf{x}{:}\mathsf{A}_{11}.\mathsf{A}_{12}$ and $\mathsf{A}_2 \stackrel{\text{whr}}{\longrightarrow}^* \Sigma \mathsf{x}{:}\mathsf{A}_{21}.\mathsf{A}_{22}$ then $\Gamma \vdash \mathsf{A}_{11} = \mathsf{A}_{21} : \mathsf{type}$ and $\Gamma, \mathsf{x}{:}\mathsf{A}_{11} \vdash \mathsf{A}_{12} = \mathsf{A}_{22} : \mathsf{type}$.

- $\Gamma \vdash \mathsf{A}_1 \approx \mathsf{A}_2 : [\![\Pi \mathsf{x}{:}\mathsf{A}.\mathsf{K}]\!]$ iff for every pair of objects M_1 and M_2 such that $\Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : \mathsf{A}$ we have $\Gamma \vdash \mathsf{A}_1\, \mathsf{M}_1 \approx \mathsf{A}_2\, \mathsf{M}_2 : [\![\mathsf{K}\,[\mathsf{M}_1/\mathsf{x}]]\!]$.
- $\Gamma \vdash \mathsf{K}_1 \approx \mathsf{K}_2 : [\![\mathsf{kind}]\!]$ iff for every pair of families A_1 and A_2, $\Gamma \vdash \mathsf{A}_1 \approx \mathsf{A}_2 : [\![\mathsf{K}_1]\!]$ iff $\Gamma \vdash \mathsf{A}_1 \approx \mathsf{A}_2 : [\![\mathsf{K}_2]\!]$.

The fundamental theorem of Logical Relations is that definitionally equal type families and kinds are logically related under all substitutions. To handle the symmetry and transitivity cases we will need that the logical relation itself is symmetric and transitive. To handle the abstraction case, we will need that the logical relation is closed under weak head expansion. To handle the family-level constant case, we will need that paths are logically related to each other. We now prove these standard lemmas before setting out to prove the fundamental lemma.

Lemma 5.2 (Paths are Logically Related) *If* $\Sigma(\mathsf{a}) = \Pi \mathsf{x}_1{:}\mathsf{A}_1 \ldots \Pi \mathsf{x}_k{:}\mathsf{A}_k.\mathsf{K}$ *(*$k \geq 0$*) and* $\Gamma \vdash \mathsf{M}_i = \mathsf{M}'_i : \mathsf{A}_i\,[\mathsf{M}_1 \ldots \mathsf{M}_{i-1}/\mathsf{x}_1 \ldots \mathsf{x}_{i-1}]$ *for all* i*, then* $\Gamma \vdash \mathsf{a}\, \mathsf{M}_1 \ldots \mathsf{M}_i \approx \mathsf{a}\, \mathsf{M}'_1 \ldots \mathsf{M}'_i : [\![\mathsf{K}\,[\mathsf{M}_1 \ldots \mathsf{M}_i/\mathsf{x}_1 \ldots \mathsf{x}_i]]\!]$.

Proof

By induction on the size of the kind K.

□

Lemma 5.3 (Closure Under Weak Head Expansion) *If* $\Gamma \vdash \mathsf{A}_{11} \approx \mathsf{A}_{21} : [\![\mathsf{K}]\!]$, $\Gamma \vdash \mathsf{A}_{12} : \mathsf{K}$, $\Gamma \vdash \mathsf{A}_{22} : \mathsf{K}$, $\mathsf{A}_{12} \stackrel{\text{whr}}{\longrightarrow}^* \mathsf{A}_{11}$ *and* $\mathsf{A}_{22} \stackrel{\text{whr}}{\longrightarrow}^* \mathsf{A}_{21}$ *then* $\Gamma \vdash \mathsf{A}_{12} \approx \mathsf{A}_{22} : [\![\mathsf{K}]\!]$.

Proof

By induction on the size of K.

□

We shall need a small fact about substituting logically related families into kinds for proving symmetry of the logical relation.

Lemma 5.4 (Equivalent Substitutions of a Valid Kind Related) *If* $\vdash \Gamma : \mathsf{ctx}$, $\Gamma \vdash \mathsf{K} : \mathsf{kind}$, $\Gamma_1 \vdash \sigma_1 = \sigma_2 : \Gamma$ *and* $\Gamma_1 \vdash \mathsf{A}_1 \approx \mathsf{A}_2 : [\![\mathsf{K}[\sigma_1]]\!]$ *then* $\Gamma_1 \vdash \mathsf{A}_1 \approx \mathsf{A}_2 : [\![\mathsf{K}[\sigma_2]]\!]$.

Proof

By induction on the size of K.

□

We need symmetry and transitivity of the relation, for the main theorem.

Lemma 5.5 (Symmetry of Logical Relation) *If* $\vdash \Gamma : \mathsf{ctx}$, $\Gamma \vdash \mathsf{K} : \mathsf{kind}$*, and* $\Gamma \vdash \mathsf{A}_1 \approx \mathsf{A}_2 : [\![\mathsf{K}]\!]$ *then* $\Gamma \vdash \mathsf{A}_2 \approx \mathsf{A}_1 : [\![\mathsf{K}]\!]$.

Proof

By induction on the size of K. We show only the product kind case.

Case 1: $\mathsf{K} = \Pi \mathsf{x}{:}\mathsf{A}.\mathsf{K}_1$

$\Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : \mathsf{A}$ — New assumption
$\Gamma \vdash \mathsf{M}_2 = \mathsf{M}_1 : \mathsf{A}$ — By rule (symmetry)
$\Gamma \vdash \mathsf{A}_1\,\mathsf{M}_2 \approx \mathsf{A}_2\,\mathsf{M}_1 : [\![\mathsf{K}_1\,[\mathsf{M}_2/\mathsf{x}]]\!]$ — By definition of relation
$\Gamma \vdash \mathsf{A}_2\,\mathsf{M}_1 \approx \mathsf{A}_1\,\mathsf{M}_2 : [\![\mathsf{K}_1\,[\mathsf{M}_2/\mathsf{x}]]\!]$ — By induction
$\Gamma \vdash \mathsf{A} : \mathsf{type}$ — By Inversion
$\vdash \Gamma, \mathsf{x}{:}\mathsf{A} : \mathsf{ctx}$ — By rule
$\Gamma \vdash \mathsf{id}_\Gamma = \mathsf{id}_\Gamma : \Gamma$ — By reflexivity of substitution equality
$\Gamma \vdash \mathsf{id}_\Gamma, \mathsf{M}_2/\mathsf{x} = \mathsf{id}_\Gamma, \mathsf{M}_1/\mathsf{x} : \Gamma, \mathsf{x}{:}\mathsf{A}$ — By Extending Substitutions by Equal Elements
$\Gamma \vdash \mathsf{A}_1\,\mathsf{M}_2 \approx \mathsf{A}_2\,\mathsf{M}_1 : [\![\mathsf{K}_1\,[\mathsf{M}_1/\mathsf{x}]]\!]$ — By Equivalent Substitutions are Related
$\Gamma \vdash \mathsf{A}_2 \approx \mathsf{A}_1 : [\![\Pi\mathsf{x}{:}\mathsf{A}.\mathsf{K}_1]\!]$ — By definition of relation

□

Lemma 5.6 (Transitivity of Logical Relation) *If* $\vdash \Gamma : \mathsf{ctx}$, $\Gamma \vdash \mathsf{K} : \mathsf{kind}$, $\Gamma \vdash \mathsf{A}_1 \approx \mathsf{A}_2 : [\![\mathsf{K}]\!]$, *and* $\Gamma \vdash \mathsf{A}_2 \approx \mathsf{A}_3 : [\![\mathsf{K}]\!]$ *then* $\Gamma \vdash \mathsf{A}_1 \approx \mathsf{A}_3 : [\![\mathsf{K}]\!]$.

Proof

By induction on the size of K.

□

We are now in a position to prove the fundamental lemma of logical relations.

Lemma 5.7 (Definitionally Equal Terms are Logically Related Under Subsitutions)

1. *If* $\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_2 : \mathsf{K}$ *and* $\Gamma_1 \vdash \sigma_1 = \sigma_2 : \Gamma$ *and* $\vdash \Gamma : \mathsf{ctx}$ *then* $\Gamma_1 \vdash \mathsf{A}_1[\sigma_1] \approx \mathsf{A}_2[\sigma_2] : [\![\mathsf{K}[\sigma_1]]\!]$.
2. *If* $\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_2 : \mathsf{kind}$ *and* $\Gamma_1 \vdash \sigma_1 = \sigma_2 : \Gamma$ *and* $\vdash \Gamma : \mathsf{ctx}$ *then* $\Gamma_1 \vdash \mathsf{A}_1[\sigma_1] \approx \mathsf{A}_2[\sigma_2] : [\![\mathsf{kind}]\!]$.

Proof

By induction on derivations. We show some of the more interesting cases.

Case 1: $$\frac{\Sigma(\mathsf{a}) = \mathsf{K}}{\Gamma \vdash \mathsf{a} = \mathsf{a} : \mathsf{K}}$$

$\cdot \vdash \mathsf{K} : \mathsf{kind}$ — By definition of valid signature
$FV(\mathsf{K}) = \cdot$ — By Free Variable Containment
$\Gamma_1 \vdash \mathsf{a} \approx \mathsf{a} : [\![\mathsf{K}]\!]$ — By Logically Related Paths
$\Gamma_1 \vdash \mathsf{a}[\sigma_1] \approx \mathsf{a}[\sigma_2] : [\![\mathsf{K}[\sigma_1]]\!]$ — By definition of substitution

Case 2: $$\frac{\Gamma \vdash \mathsf{A}_{11} = \mathsf{A}_1 : \mathsf{type} \quad \Gamma \vdash \mathsf{A}_{21} = \mathsf{A}_1 : \mathsf{type} \quad \Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{A}_{12} = \mathsf{A}_{22} : \mathsf{K}}{\Gamma \vdash \lambda\mathsf{x}{:}\mathsf{A}_{11}.\mathsf{A}_{12} = \lambda\mathsf{x}{:}\mathsf{A}_{21}.\mathsf{A}_{22} : \Pi\mathsf{x}{:}\mathsf{A}_1.\mathsf{K}}$$

$\Gamma_1 \vdash \mathsf{M}_1 = \mathsf{M}_2 : \mathsf{A}_1[\sigma_1]$ — New hypothesis
$\Gamma \vdash \mathsf{A}_1 : \mathsf{type}$ — By Regularity
$\vdash \Gamma, \mathsf{x}{:}\mathsf{A}_1 : \mathsf{ctx}$ — By rule
$\Gamma_1 \vdash \sigma_1, \mathsf{M}_1/\mathsf{x} = \sigma_2, \mathsf{M}_2/\mathsf{x} : \Gamma, \mathsf{x}{:}\mathsf{A}$ — By Extending Substitutions by Equal Elements
$\Gamma_1 \vdash \mathsf{A}_{12}[\sigma_1, \mathsf{M}_1/\mathsf{x}] \approx \mathsf{A}_{22}[\sigma_2, \mathsf{M}_2/\mathsf{x}] : [\![\mathsf{K}[\sigma_1, \mathsf{M}_1/\mathsf{x}]]\!]$ — By induction
$\Gamma_1 \vdash (\mathsf{A}_{12}\,\mathsf{M}_1)[\sigma_1] \approx (\mathsf{A}_{22}\,\mathsf{M}_2)[\sigma_2] : [\![(\mathsf{K}[\mathsf{M}_1/\mathsf{x}])[\sigma_1]]\!]$ — From previous
$\Gamma_1 \vdash (\lambda\mathsf{x}{:}\mathsf{A}_{11}[\sigma_1].\mathsf{A}_{12}[\sigma_1])\,\mathsf{M}_1 \approx (\lambda\mathsf{x}{:}\mathsf{A}_{21}[\sigma_1].\mathsf{A}_{22}[\sigma_2])\,\mathsf{M}_2 : [\![(\mathsf{K}[\mathsf{M}_1/\mathsf{x}])[\sigma_1]]\!]$ — By Closure Under Expansion
$\Gamma_1 \vdash \lambda\mathsf{x}{:}\mathsf{A}_{11}[\sigma_1].\mathsf{A}_{12}[\sigma_1] \approx \lambda\mathsf{x}{:}\mathsf{A}_{21}[\sigma_1].\mathsf{A}_{22}[\sigma_2] : [\![\Pi\mathsf{x}{:}\mathsf{A}.\mathsf{K}]\!]$ — By definition of relation

Case 3: $$\frac{\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_2 : \Pi \mathsf{x}{:}\mathsf{A}_3.\mathsf{K} \qquad \Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : \mathsf{A}_3}{\Gamma \vdash \mathsf{A}_1\,\mathsf{M}_1 = \mathsf{A}_2\,\mathsf{M}_2 : \mathsf{K}\,[\mathsf{M}_1/\mathsf{x}]}$$

$\Gamma_1 \vdash \mathsf{A}_1[\sigma_1] \approx \mathsf{A}_2[\sigma_2] : [\![\Pi \mathsf{x}{:}\mathsf{A}_3[\sigma_1].\mathsf{K}[\sigma_1]]\!]$ By induction
$\Gamma_1 \vdash \mathsf{M}_1[\sigma_1] = \mathsf{M}_2[\sigma_2] : \mathsf{A}_3[\sigma_1]$ By Functionality of Equality
$\Gamma_1 \vdash (\mathsf{A}_1[\sigma_1])\,(\mathsf{M}_1[\sigma_1]) \approx (\mathsf{A}_2[\sigma_2])\,(\mathsf{M}_2[\sigma_2]) : [\![(\mathsf{K}\,[LFobj_1[\sigma_1]/\mathsf{x}])[\sigma_1]]\!]$ By definition of relation
$\Gamma_1 \vdash (\mathsf{A}_1\,\mathsf{M}_1)[\sigma_1] \approx (\mathsf{A}_2\,\mathsf{M}_2)[\sigma_2] : [\![(\mathsf{K}\,[LFobj_1[\sigma_1]/\mathsf{x}])[\sigma_1]]\!]$ By definition of substitution

Case 4: $$\frac{\Gamma \vdash \mathsf{A} : \mathsf{type} \qquad \Gamma, \mathsf{x}{:}\mathsf{A} \vdash \mathsf{A}_1 = \mathsf{A}_2 : \mathsf{K} \qquad \Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : \mathsf{A}}{\Gamma \vdash (\lambda \mathsf{x}{:}\mathsf{A}.\mathsf{A}_1)\,\mathsf{M}_1 = \mathsf{A}_2\,[\mathsf{M}_2/\mathsf{x}] : \mathsf{K}\,[\mathsf{M}_1/\mathsf{x}]}$$

$\Gamma_1 \vdash \mathsf{M}_1[\sigma_1] = \mathsf{M}_2[\sigma_2] : \mathsf{A}[\sigma_1]$ By Functionality of Equality
$\Gamma \vdash \mathsf{A} : \mathsf{type}$ By assumption
$\vdash \Gamma, \mathsf{x}{:}\mathsf{A} : \mathsf{ctx}$ By rule
$\Gamma_1 \vdash \sigma_1, \mathsf{M}_1[\sigma_1]/\mathsf{x} = \sigma_2, \mathsf{M}_2[\sigma_2]/\mathsf{x} : \Gamma, \mathsf{x}{:}\mathsf{A}$ By Extending Substitutions by Equal Elements
$\Gamma_1 \vdash \mathsf{A}_1\,[\mathsf{M}_1[\sigma_1]/\mathsf{x}]\,[\sigma_1] \approx \mathsf{A}_2\,[\mathsf{M}_2[\sigma_2]/\mathsf{x}]\,[\sigma_2] : [\![\mathsf{K}\,[\mathsf{M}_1[\sigma_1]/\mathsf{x}]\,[\sigma_1]]\!]$ By induction
$\Gamma_1 \vdash (\mathsf{A}_1[\sigma_1])\,[\mathsf{M}_1[\sigma_1]/\mathsf{x}] \approx (\mathsf{A}_2[\sigma_2])\,[\mathsf{M}_2[\sigma_2]/\mathsf{x}] : [\![(\mathsf{K}[\sigma_1])\,[\mathsf{M}_1[\sigma_1]/\mathsf{x}]]\!]$ From previous
$\Gamma_1 \vdash (\lambda \mathsf{x}{:}\mathsf{A}[\sigma_1].\mathsf{A}_1[\sigma_1])\,\mathsf{M}_1[\sigma_1] \approx (\mathsf{A}_2[\sigma_2])\,[\mathsf{M}_2[\sigma_2]/\mathsf{x}] : [\![(\mathsf{K}[\sigma_1])\,[\mathsf{M}_1[\sigma_1]/\mathsf{x}]]\!]$ By Closure Under Expansion

Case 5: $$\frac{\Gamma \vdash \mathsf{A}_{11} = \mathsf{A}_{21} : \mathsf{type} \qquad \Gamma \vdash \mathsf{A}_{11} : \mathsf{type} \qquad \Gamma, \mathsf{x}{:}\mathsf{A}_{11} \vdash \mathsf{A}_{12} = \mathsf{A}_{22} : \mathsf{type}}{\Gamma \vdash \Sigma \mathsf{x}{:}\mathsf{A}_{11}.\mathsf{A}_{12} = \Sigma \mathsf{x}{:}\mathsf{A}_{21}.\mathsf{A}_{22} : \mathsf{type}}$$

$\Gamma \xrightarrow{\text{whr}} \Sigma \mathsf{x}{:}\mathsf{A}_{11}[\sigma_1].\mathsf{A}_{12}[\sigma_1]\mathsf{A}_1$ implies $\mathsf{A}_1 = \Sigma \mathsf{x}{:}\mathsf{A}_{11}[\sigma_1].\mathsf{A}_{12}[\sigma_1]$ By definition of reduction
$\Gamma \xrightarrow{\text{whr}} \Sigma \mathsf{x}{:}\mathsf{A}_{21}[\sigma_2].\mathsf{A}_{22}[\sigma_2]\mathsf{A}_2$ implies $\mathsf{A}_2 = \Sigma \mathsf{x}{:}\mathsf{A}_{21}[\sigma_2].\mathsf{A}_{22}[\sigma_2]$ By definition of reduction
$\Gamma_1 \vdash \mathsf{A}_{11}[\sigma_1] = \mathsf{A}_{21}[\sigma_2] : \mathsf{type}$ By Functionality of Equality
$\Gamma_1 \vdash \mathsf{A}_{11}[\sigma_1] : \mathsf{type}$ By Regularity
$\vdash \Gamma_1, \mathsf{x}{:}\mathsf{A}_{11}[\sigma_1] : \mathsf{ctx}$ By rule
$\Gamma \vdash \mathsf{A}_{11} : \mathsf{type}$ By Regularity
$\vdash \Gamma, \mathsf{x}{:}\mathsf{A}_{11} : \mathsf{ctx}$ By rule
$\Gamma_1, \mathsf{x}{:}\mathsf{A}_{11}[\sigma_1] \vdash \sigma_1, u/u = \sigma_2, u/u : \Gamma, \mathsf{x}{:}\mathsf{A}_{11}$ By Extending Substitutions
$\Gamma_1, \mathsf{x}{:}\mathsf{A}_{11} \vdash \mathsf{A}_{12}[\sigma_1, u/u] : \mathsf{A}_{22}[\sigma_2, u/u]\mathsf{type}$ By Functionality
$\Gamma_1, \mathsf{x}{:}\mathsf{A}_{11} \vdash \mathsf{A}_{12}[\sigma_1] : \mathsf{A}_{22}[\sigma_2]\mathsf{type}$ From previous
$\Gamma_1 \vdash \Sigma \mathsf{x}{:}\mathsf{A}_{11}.\mathsf{A}_{12} \approx \Sigma \mathsf{x}{:}\mathsf{A}_{21}.\mathsf{A}_{22} : [\![\mathsf{type}]\!]$ By definition of relation

Case 6: $$\frac{\Gamma \vdash \mathsf{A}_2 = \mathsf{A}_1 : \mathsf{K}}{\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_2 : \mathsf{K}}$$

$\Gamma_1 \vdash \sigma_2 = \sigma_1 : \Gamma$ By symmetry of substitution equality
$\Gamma_1 \vdash \mathsf{A}_2[\sigma_2] \approx \mathsf{A}_1[\sigma_1] : [\![\mathsf{K}[\sigma_2]]\!]$ By induction
$\Gamma_1 \vdash \mathsf{A}_1[\sigma_1] \approx \mathsf{A}_2[\sigma_2] : [\![\mathsf{K}[\sigma_2]]\!]$ By Symmetry of Relation
$\Gamma_1 \vdash \mathsf{A}_1[\sigma_1] \approx \mathsf{A}_2[\sigma_2] : [\![\mathsf{K}[\sigma_1]]\!]$ By Relation of Equal Substitutions

Case 7: $$\frac{\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_2 : \mathsf{K} \qquad \Gamma \vdash \mathsf{A}_2 = \mathsf{A}_3 : \mathsf{K}}{\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_3 : \mathsf{K}}$$

$\Gamma_1 \vdash \sigma_2 = \sigma_1 : \Gamma$ By symmetry of substitution equality
$\Gamma_1 \vdash \sigma_1 = \sigma_1 : \Gamma$ By transitivity of substitution equality
$\Gamma_1 \vdash \mathsf{A}_1[\sigma_1] \approx \mathsf{A}_2[\sigma_1] : [\![\mathsf{K}[\sigma_1]]\!]$ By induction
$\Gamma_1 \vdash \mathsf{A}_2[\sigma_1] \approx \mathsf{A}_3[\sigma_2] : [\![\mathsf{K}[\sigma_1]]\!]$ By induction
$\Gamma_1 \vdash \mathsf{A}_1[\sigma_1] \approx \mathsf{A}_3[\sigma_2] : [\![\mathsf{K}[\sigma_1]]\!]$ By Transitivity of Relation

Case 8: $$\frac{\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_2 : \mathsf{type} \qquad \Gamma \vdash \mathsf{A}_1 : \mathsf{type} \qquad \Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{K}_1 = \mathsf{K}_2 : \mathsf{kind}}{\Gamma \vdash \Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{K}_1 = \Pi \mathsf{x}{:}\mathsf{A}_2.\mathsf{K}_2 : \mathsf{kind}}$$

The definition of relation at product kinds involves checking a bi-implication. We show one direction, the other is similar.

$\Gamma_1 \vdash \mathsf{A}_3 \approx \mathsf{A}_4 : [\![\Pi \mathsf{x}{:}\mathsf{A}_2[\sigma_2].\mathsf{K}_2[\sigma_2]]\!]$ New hypothesis
$\Gamma_1 \vdash \mathsf{M}_1 = \mathsf{M}_2 : \mathsf{A}_1[\sigma_1]$ New hypothesis
$\Gamma_1 \vdash \mathsf{A}_1[\sigma_1] = \mathsf{A}_2[\sigma_2] : \mathsf{type}$ By Functionality of Equality
$\Gamma_1 \vdash \mathsf{M}_1 = \mathsf{M}_2 : \mathsf{A}_2[\sigma_2]$ By rule (type conversion)
$\Gamma_1 \vdash \mathsf{A}_3\,\mathsf{M}_1 \approx \mathsf{A}_4\,\mathsf{M}_2 : [\![\mathsf{K}_2[\sigma_2]\,[\mathsf{M}_1/\mathsf{x}]]\!]$ By definition of relation
$\Gamma_1 \vdash \sigma_1, \mathsf{M}_1/\mathsf{x} = \sigma_2, \mathsf{M}_1/\mathsf{x} : \Gamma, \mathsf{x}{:}\mathsf{A}$ By Extending Substitution
$\Gamma_1 \vdash \mathsf{K}_1[\sigma_1, \mathsf{M}_1/\mathsf{x}] \approx \mathsf{K}_2[\sigma_2, \mathsf{M}_1/\mathsf{x}] : [\![\mathsf{kind}]\!]$ By induction
$\Gamma_1 \vdash \mathsf{A}_3\,\mathsf{M}_1 \approx \mathsf{A}_4\,\mathsf{M}_2 : [\![\mathsf{K}_1[\sigma_1]\,[\mathsf{M}_1/\mathsf{x}]]\!]$ By definition of relation
$\Gamma_1 \vdash \mathsf{A}_3 \approx \mathsf{A}_4 : [\![\Pi \mathsf{x}{:}\mathsf{A}_1[\sigma_1].\mathsf{K}_1[\sigma_1]]\!]$ By definition of relation

□

We can now prove the injectivity property we are interested in.

Theorem 5.8 (Injectivity of Products and Sums)

1. *If* $\Gamma \vdash \Pi \mathsf{x}{:}\mathsf{A}_{11}.\mathsf{A}_{12} = \Pi \mathsf{x}{:}\mathsf{A}_{21}.\mathsf{A}_{22} : \mathsf{type}$ *then* $\Gamma \vdash \mathsf{A}_{11} = \mathsf{A}_{21} : \mathsf{type}$ *and* $\Gamma, \mathsf{x}{:}\mathsf{A}_{11} \vdash \mathsf{A}_{12} = \mathsf{A}_{22} : \mathsf{type}$.
2. *If* $\Gamma \vdash \Sigma \mathsf{x}{:}\mathsf{A}_{11}.\mathsf{A}_{12} = \Sigma \mathsf{x}{:}\mathsf{A}_{21}.\mathsf{A}_{22} : \mathsf{type}$ *then* $\Gamma \vdash \mathsf{A}_{11} = \mathsf{A}_{21} : \mathsf{type}$ *and* $\Gamma, \mathsf{x}{:}\mathsf{A}_{11} \vdash \mathsf{A}_{12} = \mathsf{A}_{22} : \mathsf{type}$.

Proof

Direct, from the previous lemma and definition of the relation. We show one case, the other is analogous

Case 1: $\Gamma \vdash \Pi \mathsf{x}{:}\mathsf{A}_{11}.\mathsf{A}_{12} = \Pi \mathsf{x}{:}\mathsf{A}_{21}.\mathsf{A}_{22} : \mathsf{type}$

$\Gamma \vdash \mathsf{id}_\Gamma = \mathsf{id}_\Gamma : \Gamma$ By reflexivity of substitutions
$\Gamma \vdash (\Pi \mathsf{x}{:}\mathsf{A}_{11}.\mathsf{A}_{12})[\mathsf{id}_\Gamma] \approx (\Pi \mathsf{x}{:}\mathsf{A}_{21}.\mathsf{A}_{22})[\mathsf{id}_\Gamma] : [\![\mathsf{type}]\!]$ By Definitionally Equal Terms Related
$\Gamma \vdash \Pi \mathsf{x}{:}\mathsf{A}_{11}.\mathsf{A}_{12} \approx \Pi \mathsf{x}{:}\mathsf{A}_{21}.\mathsf{A}_{22} : [\![\mathsf{type}]\!]$ From previous
$\Gamma \vdash \mathsf{A}_{11} = \mathsf{A}_{21} : \mathsf{type}, \Gamma, \mathsf{x}{:}\mathsf{A}_{11} \vdash \mathsf{A}_{12} = \mathsf{A}_{22} : \mathsf{type}$ By definition of relation

□

6 Equality Algorithm

An algorithm for deciding equality is given following Harper and Pfenning [5]. This uses "erased" types and kinds to direct the comparison. The problem with having unerased types and kinds is that since we have dependent types, we have to pick some object to substitute in for the application or projection cases. Then it is not obvious that the algorithm is symmetric, or transitive. Fortunately, the comparison depends only on the shape of the type or kind, which is what the erasure function gets at.

6.1 Erasure

We erase all dependencies in families and kinds for the algorithm. This process identifies types that differ only in the terms appearing in them.

Simple Kinds	$\kappa ::=$	type^-	simple kind of types
	$\mid$	$\tau \to \kappa$	simple arrow kind
Simple Types	$\tau ::=$	α	simple type constants
	$\mid$	$\tau_1 \to \tau_2$	simple arrow type
	$\mid$	$\tau_1 \times \tau_2$	simple product type
Simple Contexts	$\Delta ::=$	$\cdot$	empty
	$\mid$	$\Delta, \mathsf{x}{:}\tau$	context extension

The erasure function is defined as follows:

$$\begin{array}{rcl} (\mathsf{a})^- & = & \alpha \\ (\lambda \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2)^- & = & \mathsf{A}_2{}^- \\ (\mathsf{A}\,\mathsf{M})^- & = & \mathsf{A}^- \\ (\Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2)^- & = & \mathsf{A}_1{}^- \to \mathsf{A}_2{}^- \\ (\Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2)^- & = & \mathsf{A}_1{}^- \times \mathsf{A}_2{}^- \\ (1)^- & = & 1^- \end{array} \qquad \begin{array}{rcl} (\mathsf{type})^- & = & \mathsf{type}^- \\ (\Pi \mathsf{x}{:}\mathsf{A}.\mathsf{K})^- & = & \mathsf{A}^- \to \mathsf{K}^- \\ \\ (\cdot)^- & = & \cdot \\ (\Gamma, \mathsf{x}{:}\mathsf{A})^- & = & \Gamma^-, \mathsf{x}{:}\mathsf{A}^- \end{array}$$

We will need some basic facts about erasure. We notice that erasure eliminates dependencies on terms, and further, for type-level abstractions and eliminations, erasure returns just the head family. Given this informal description, it is not surprising that erasure is invariant on substitution, and further, definitionally equal types and kinds erase to the same simple type and kind.

Lemma 6.1 (Erasure Preservation)

1. *For any* A*,* x *and* M*,* $(\mathsf{A}\,[\mathsf{M}/\mathsf{x}])^- = \mathsf{A}^-$*.*
2. *For any* K*,* x *and* M*,* $(\mathsf{K}\,[\mathsf{M}/\mathsf{x}])^- = \mathsf{K}^-$*.*

Proof
By induction on the structure of the type family or kind in the premise.

□

Lemma 6.2 (Erasure Preservation of Equality)

1. *If* $\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_2 : \mathsf{K}$ *then* $\mathsf{A}_1{}^- = \mathsf{A}_2{}^-$
2. *If* $\Gamma \vdash \mathsf{K}_1 = \mathsf{K}_2 : \mathsf{kind}$ *then* $\mathsf{K}_1{}^- = \mathsf{K}_2{}^-$

Proof
By induction on the derivation of equality. □

Since we will case analyze on erased types, we need to say something about what type families erase to a particular erased type. We notice that a variety of equal types can erase to the same erased type, but fortunately, all of these have the same weak head normal form.

Lemma 6.3

1. *If* $\Gamma \vdash \mathsf{A} : \mathsf{type}$*,* $\mathsf{A}^- = 1^-$ *and* $\mathsf{A} \stackrel{\text{whr}}{\not\longrightarrow}$*, then* $\mathsf{A} = 1$*.*
2. *If* $\Gamma \vdash \mathsf{A} : \mathsf{type}$*,* $\mathsf{A}^- = \tau_1 \to \tau_2$ *and* $\mathsf{A} \stackrel{\text{whr}}{\not\longrightarrow}$*, then* $\mathsf{A} = \Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2$*.*
3. *If* $\Gamma \vdash \mathsf{A} : \mathsf{type}$*,* $\mathsf{A}^- = \tau_1 \times \tau_2$ *and* $\mathsf{A} \stackrel{\text{whr}}{\not\longrightarrow}$*, then* $\mathsf{A} = \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2$*.*

Proof
By inspection of the construction of A.

□

6.2 Algorithmic Equality

We are now in a position to give the algorithm. This is given by five mututally recursive judgments. The idea of the algorithm is that extensionality is used whenever terms are compared at non-atomic types, to drive the type at which comparison is done to an atomic type. Once at atomic type, the comparison is done by weak-head normalizing both sides and comparing structurally. The structural phase compares the primitive head of the term, and uses the type-directed phase for terms down the spine. An analogous process is carried out at the level of type families.

6.2.1 Judgements

$\Delta \vdash \mathsf{M}_1 \Longleftrightarrow \mathsf{M}_2 : \tau$	Type Directed Object Equality
$\Delta \vdash \mathsf{M}_1 \longleftrightarrow \mathsf{M}_2 : \tau$	Structural Object Equality
$\Delta \vdash \tau_1 \Longleftrightarrow \tau_2 : \kappa$	Kind Directed Type Equality
$\Delta \vdash \tau_1 \longleftrightarrow \tau_2 : \kappa$	Structural Type Equality
$\Delta \vdash \kappa_1 \longleftrightarrow \kappa_2 : \mathsf{kind}^-$	Structural Kind Equality

6.3 Inference Rules

$\boxed{\Delta \vdash \mathsf{M}_1 \Longleftrightarrow \mathsf{M}_2 : \tau}$

$$\frac{\mathsf{M}_1 \stackrel{\text{whr}}{\longrightarrow} \mathsf{M}_2 \quad \Delta \vdash \mathsf{M}_2 \Longleftrightarrow \mathsf{M} : \alpha}{\Delta \vdash \mathsf{M}_1 \Longleftrightarrow \mathsf{M} : \alpha} \qquad \frac{\mathsf{M}_1 \stackrel{\text{whr}}{\longrightarrow} \mathsf{M}_2 \quad \Delta \vdash \mathsf{M} \Longleftrightarrow \mathsf{M}_2 : \alpha}{\Delta \vdash \mathsf{M} \Longleftrightarrow \mathsf{M}_1 : \alpha} \qquad \frac{\Delta \vdash \mathsf{M}_1 \longleftrightarrow \mathsf{M}_2 : \alpha}{\Delta \vdash \mathsf{M}_1 \Longleftrightarrow \mathsf{M}_2 : \alpha}$$

$$\frac{\Delta, \mathsf{x}{:}\tau_1 \vdash \mathsf{M}_1\,\mathsf{x} \Longleftrightarrow \mathsf{M}_2\,\mathsf{x} : \tau_2}{\Delta \vdash \mathsf{M}_1 \Longleftrightarrow \mathsf{M}_2 : \tau_1 \to \tau_2} \qquad \frac{\Delta \vdash \pi_1\mathsf{M}_1 \Longleftrightarrow \pi_1\mathsf{M}_2 : \tau_1 \quad \Delta \vdash \pi_2\mathsf{M}_1 \Longleftrightarrow \pi_2\mathsf{M}_2 : \tau_2}{\Delta \vdash \mathsf{M}_1 \Longleftrightarrow \mathsf{M}_2 : \tau_1 \times \tau_2}$$

$$\frac{}{\Delta \vdash \mathsf{M}_1 \Longleftrightarrow \mathsf{M}_2 : 1}$$

$\boxed{\Delta \vdash \mathsf{M}_1 \longleftrightarrow \mathsf{M}_2 : \tau}$

$$\frac{\Sigma(\mathsf{c}) = \mathsf{A}}{\Delta \vdash \mathsf{c} \longleftrightarrow \mathsf{c} : \mathsf{A}^-} \qquad \frac{\Delta(\mathsf{x}) = \tau}{\Delta \vdash \mathsf{x} \longleftrightarrow \mathsf{x} : \tau} \qquad \frac{\Delta \vdash \mathsf{M}_{11} \longleftrightarrow \mathsf{M}_{21} : \tau_2 \to \tau_1 \quad \Delta \vdash \mathsf{M}_{12} \Longleftrightarrow \mathsf{M}_{22} : \tau_2}{\Delta \vdash \mathsf{M}_{11}\,\mathsf{M}_{12} \longleftrightarrow \mathsf{M}_{21}\,\mathsf{M}_{22} : \tau_1}$$

$$\frac{\Delta \vdash \mathsf{M}_1 \longleftrightarrow \mathsf{M}_2 : \tau_1 \times \tau_2}{\Delta \vdash \pi_i\mathsf{M}_1 \longleftrightarrow \pi_i\mathsf{M}_2 : \tau_i}$$

$\boxed{\Delta \vdash \mathsf{A}_1 \Longleftrightarrow \mathsf{A}_2 : \kappa}$

$$\frac{\mathsf{A}_1 \stackrel{\text{whr}}{\longrightarrow} \mathsf{A}_2 \quad \Delta \vdash \mathsf{A}_2 \Longleftrightarrow \mathsf{A} : \mathsf{type}^-}{\Delta \vdash \mathsf{A}_1 \Longleftrightarrow \mathsf{A} : \mathsf{type}^-} \qquad \frac{\mathsf{A}_1 \stackrel{\text{whr}}{\longrightarrow} \mathsf{A}_2 \quad \Delta \vdash \mathsf{A} \Longleftrightarrow \mathsf{A}_2 : \mathsf{type}^-}{\Delta \vdash \mathsf{A} \Longleftrightarrow \mathsf{A}_1 : \mathsf{type}^-} \qquad \frac{\Delta \vdash \mathsf{A}_1 \longleftrightarrow \mathsf{A}_2 : \mathsf{type}^-}{\Delta \vdash \mathsf{A}_1 \Longleftrightarrow \mathsf{A}_2 : \mathsf{type}^-}$$

$$\frac{\Delta, \mathsf{x}{:}\tau \vdash \mathsf{A}_1\,\mathsf{x} \Longleftrightarrow \mathsf{A}_2\,\mathsf{x} : \kappa}{\Delta \vdash \mathsf{A}_1 \Longleftrightarrow \mathsf{A}_2 : \tau \to \kappa}$$

$\boxed{\Delta \vdash \mathsf{A}_1 \longleftrightarrow \mathsf{A}_2 : \kappa}$

$$\frac{\Sigma(\mathsf{a}) = \mathsf{K}}{\Delta \vdash \mathsf{a} \longleftrightarrow \mathsf{a} : \mathsf{K}^-} \qquad \frac{}{\Delta \vdash 1 \longleftrightarrow 1 : \mathsf{type}^-} \qquad \frac{\Delta \vdash \mathsf{A}_1 \longleftrightarrow \mathsf{A}_2 : \tau \to \kappa \quad \Delta \vdash \mathsf{M}_1 \Longleftrightarrow \mathsf{M}_2 : \tau}{\Delta \vdash \mathsf{A}_1\,\mathsf{M}_1 \longleftrightarrow \mathsf{A}_2\,\mathsf{M}_2 : \kappa}$$

$$\frac{\Delta \vdash \mathsf{A}_{11} \Longleftrightarrow \mathsf{A}_{21} : \mathsf{type}^- \quad \Delta, \mathsf{x}{:}\mathsf{A}_{11}{}^- \vdash \mathsf{A}_{12} \Longleftrightarrow \mathsf{A}_{22} : \mathsf{type}^-}{\Delta \vdash \Pi\mathsf{x}{:}\mathsf{A}_{11}.\mathsf{A}_{12} \longleftrightarrow \Pi\mathsf{x}{:}\mathsf{A}_{21}.\mathsf{A}_{22} : \mathsf{type}^-}$$

$$\frac{\Delta \vdash \mathsf{A}_{11} \Longleftrightarrow \mathsf{A}_{21} : \mathsf{type}^- \quad \Delta, \mathsf{x}{:}\mathsf{A}_{11}{}^- \vdash \mathsf{A}_{12} \Longleftrightarrow \mathsf{A}_{22} : \mathsf{type}^-}{\Delta \vdash \Sigma\mathsf{x}{:}\mathsf{A}_{11}.\mathsf{A}_{12} \longleftrightarrow \Sigma\mathsf{x}{:}\mathsf{A}_{21}.\mathsf{A}_{22} : \mathsf{type}^-}$$

$\boxed{\Delta \vdash \mathsf{K}_1 \longleftrightarrow \mathsf{K}_2 : \mathsf{kind}^-}$

$$\frac{}{\Delta \vdash \mathsf{type} \longleftrightarrow \mathsf{type} : \mathsf{kind}^-} \qquad \frac{\Delta \vdash \mathsf{A}_1 \Longleftrightarrow \mathsf{A}_2 : \mathsf{type}^- \quad \Delta, \mathsf{x}{:}\mathsf{A}_1{}^- \vdash \mathsf{K}_1 \longleftrightarrow \mathsf{K}_2 : \mathsf{kind}^-}{\Delta \vdash \Pi\mathsf{x}{:}\mathsf{A}_1.\mathsf{K}_1 \longleftrightarrow \Pi\mathsf{x}{:}\mathsf{A}_2.\mathsf{K}_2 : \mathsf{kind}^-}$$

We will show this algorithm to be sound and complete for the equality judgments. We will need some basic facts about the algorithm, such as that weakening of the context is allowed, and the fact that the algorithm is deterministic.

Lemma 6.4 (Weakening of Algorithmic Equality) *For all algorithmic judgments $\mathcal{J}$, if $\Delta \vdash \mathcal{J}$ and $\Delta \subseteq \Delta^+$, then $\Delta^+ \vdash \mathcal{J}$.*

Proof

By induction on the first judgment.

□

Lemma 6.5 (Determinacy of Algorithmic Equality)

1. *If* $\Delta \vdash \mathsf{M}_1 \longleftrightarrow \mathsf{M}_2 : \tau$ *then* $\mathsf{M}_1 \overset{\text{whr}}{\not\longrightarrow}$.
2. *If* $\Delta \vdash \mathsf{M}_1 \longleftrightarrow \mathsf{M}_2 : \tau$ *then* $\mathsf{M}_2 \overset{\text{whr}}{\not\longrightarrow}$.
3. *If* $\Delta \vdash \mathsf{M}_1 \longleftrightarrow \mathsf{M}_2 : \tau_1$ *and* $\Delta \vdash \mathsf{M}_1 \longleftrightarrow \mathsf{M}_3 : \tau_2$ *then* $\tau_1 = \tau_2$.
4. *If* $\Delta \vdash \mathsf{A}_1 \longleftrightarrow \mathsf{A}_2 : \kappa$ *then* $\mathsf{A}_1 \overset{\text{whr}}{\not\longrightarrow}$.
5. *If* $\Delta \vdash \mathsf{A}_1 \longleftrightarrow \mathsf{A}_2 : \kappa$ $\mathsf{A}_2 \overset{\text{whr}}{\not\longrightarrow}$.
6. *If* $\Delta \vdash \mathsf{A}_1 \longleftrightarrow \mathsf{A}_2 : \kappa_1$ *and* $\Delta \vdash \mathsf{A}_1 \longleftrightarrow \mathsf{A}_3 : \kappa_2$ *then* $\kappa_1 = \kappa_2$.

Proof

By induction on the derivation of the premise.

□

Lemma 6.6 (Erasure Preservation of Algorithmic Equality)

1. *If* $\Delta \vdash \mathsf{A}_1 \Longleftrightarrow \mathsf{A}_2 : \kappa$ *then* $\mathsf{A}_1{}^- = \mathsf{A}_2{}^-$.
2. *If* $\Delta \vdash \mathsf{A}_1 \longleftrightarrow \mathsf{A}_2 : \kappa$ *then* $\mathsf{A}_1{}^- = \mathsf{A}_2{}^-$.

3. *If* $\mathsf{A}_1 \xrightarrow{\text{whr}} \mathsf{A}_2$ *then* $\mathsf{A}_1{}^- = \mathsf{A}_2{}^-$.

Proof

By induction on the derivation.

□

To prove that the algorithm is complete, in the cases for the use of symmetry and transitivity rules, we will need that the algorithm itself is symmetric and transitive.

Lemma 6.7 (Symmetry of Algorithmic Equality)

1. *If* $\Delta \vdash \mathsf{M}_1 \Longleftrightarrow \mathsf{M}_2 : \tau$ *then* $\Delta \vdash \mathsf{M}_2 \Longleftrightarrow \mathsf{M}_1 : \tau$.
2. *If* $\Delta \vdash \mathsf{M}_1 \longleftrightarrow \mathsf{M}_2 : \tau$ *then* $\Delta \vdash \mathsf{M}_2 \Longleftrightarrow \mathsf{M}_1 : \tau$.
3. *If* $\Delta \vdash \mathsf{A}_1 \Longleftrightarrow \mathsf{A}_2 : \kappa$ *then* $\Delta \vdash \mathsf{A}_2 \Longleftrightarrow \mathsf{A}_1 : \kappa$.
4. *If* $\Delta \vdash \mathsf{A}_1 \longleftrightarrow \mathsf{A}_2 : \kappa$ *then* $\Delta \vdash \mathsf{A}_2 \longleftrightarrow \mathsf{A}_1 : \kappa$.
5. *If* $\Delta \vdash \mathsf{K}_1 \longleftrightarrow \mathsf{K}_2 : \mathsf{kind}^-$ *then* $\Delta \vdash \mathsf{K}_2 \longleftrightarrow \mathsf{K}_1 : \mathsf{kind}^-$.

Proof

By induction on the derivation.

□

Lemma 6.8 (Transitivity of Algorithmic Equality)

1. *If* $\Delta \vdash \mathsf{M}_1 \Longleftrightarrow \mathsf{M}_2 : \tau$ *and* $\Delta \vdash \mathsf{M}_2 \Longleftrightarrow \mathsf{M}_3 : \tau$*, then* $\Delta \vdash \mathsf{M}_1 \Longleftrightarrow \mathsf{M}_3 : \tau$.
2. *If* $\Delta \vdash \mathsf{M}_1 \longleftrightarrow \mathsf{M}_2 : \tau$ *and* $\Delta \vdash \mathsf{M}_2 \longleftrightarrow \mathsf{M}_3 : \tau$*, then* $\Delta \vdash \mathsf{M}_1 \longleftrightarrow \mathsf{M}_3 : \tau$.
3. *If* $\Delta \vdash \mathsf{A}_1 \Longleftrightarrow \mathsf{A}_2 : \kappa$ *and* $\Delta \vdash \mathsf{A}_2 \Longleftrightarrow \mathsf{A}_3 : \kappa$*, then* $\Delta \vdash \mathsf{A}_1 \Longleftrightarrow \mathsf{A}_3 : \kappa$.
4. *If* $\Delta \vdash \mathsf{A}_1 \longleftrightarrow \mathsf{A}_2 : \kappa$ *and* $\Delta \vdash \mathsf{A}_2 \longleftrightarrow \mathsf{A}_3 : \kappa$*, then* $\Delta \vdash \mathsf{A}_1 \longleftrightarrow \mathsf{A}_3 : \kappa$.
5. *If* $\Delta \vdash \mathsf{K}_1 \longleftrightarrow \mathsf{K}_2 : \mathsf{kind}^-$ *and* $\Delta \vdash \mathsf{K}_2 \longleftrightarrow \mathsf{K}_3 : \mathsf{kind}^-$*, then* $\Delta \vdash \mathsf{K}_1 \longleftrightarrow \mathsf{K}_3 : \mathsf{kind}^-$.

Proof

By induction on the two judgments in the premises.

□

7 Completeness of the Algorithm

The completeness of the algorithm is proved by means of a Kripke logical relation.

7.1 A Kripke Logical Relation

The logical relation we use is defined by induction on the simple types and kinds.

- $\Delta \vdash \mathsf{M}_1$ **is** M_2 **in** $[\![\alpha]\!]$ iff $\Delta \vdash \mathsf{M}_1 \Longleftrightarrow \mathsf{M}_2 : \alpha$.
- $\Delta \vdash \mathsf{M}_1$ **is** M_2 **in** $[\![\tau_1 \to \tau_2]\!]$ iff for every context Δ_1 with $\Delta \subseteq \Delta_1$ and every pair of objects M_1' and M_2' such that $\Delta_1 \vdash \mathsf{M}_1'$ **is** M_2' **in** $[\![\tau_1]\!]$ we have $\Delta_1 \vdash \mathsf{M}_1\,\mathsf{M}_1'$ **is** $\mathsf{M}_2\,\mathsf{M}_2'$ **in** $[\![\tau_2]\!]$.
- $\Delta \vdash \mathsf{M}_1$ **is** M_2 **in** $[\![\tau_1 \times \tau_2]\!]$ iff $\Delta \vdash \pi_1\mathsf{M}_1$ **is** $\pi_1\mathsf{M}_2$ **in** $[\![\tau_1]\!]$ and $\Delta \vdash \pi_1\mathsf{M}_2$ **is** $\pi_2\mathsf{M}_2$ **in** $[\![\tau_2]\!]$.
- $\Delta \vdash \mathsf{M}_1$ **is** M_2 **in** $[\![1^-]\!]$ always.
- $\Delta \vdash \mathsf{A}_1$ **is** A_2 **in** $[\![\mathsf{type}^-]\!]$ iff $\Delta \vdash \mathsf{A}_1 \Longleftrightarrow \mathsf{A}_2 : \mathsf{type}^-$.
- $\Delta \vdash \mathsf{A}_1$ **is** A_2 **in** $[\![\tau \to \kappa]\!]$ iff for every context Δ_1 with $\Delta \subseteq \Delta_1$ and every pair of objects M_1 and M_2 such that $\Delta_1 \vdash \mathsf{M}_1$ **is** M_2 **in** $[\![\tau]\!]$ we have $\Delta_1 \vdash \mathsf{A}_1\,\mathsf{M}_1$ **is** $\mathsf{A}_2\,\mathsf{M}_2$ **in** $[\![\kappa]\!]$.
- $\Delta \vdash \sigma_1$ **is** σ_2 **in** $[\![\cdot]\!]$ iff $\sigma_1 = \sigma_2 = \cdot$.
- $\Delta_1 \vdash \sigma_1\,[\mathsf{M}_1/\mathsf{x}]$ **is** $\sigma_2\,[\mathsf{M}_2/\mathsf{x}]$ **in** $[\![\Delta, \mathsf{x}{:}\tau]\!]$ iff $\Delta_1 \vdash \sigma_1$ **is** σ_2 **in** $[\![\Delta]\!]$ and $\Delta_1 \vdash \mathsf{M}_1$ **is** M_2 **in** $[\![\tau]\!]$.

Lemma 7.1 (Logically Related Terms are Algorithmically Equal [Fundamental Lemma])

1. *If* $\Delta \vdash \mathsf{M}_1$ **is** M_2 **in** $[\![\tau]\!]$ *then* $\Delta \vdash \mathsf{M}_1 \Longleftrightarrow \mathsf{M}_2 : \tau$.
2. *If* $\Delta \vdash \mathsf{A}_1$ **is** A_2 **in** $[\![\kappa]\!]$ *then* $\Delta \vdash \mathsf{A}_1 \Longleftrightarrow \mathsf{A}_2 : \kappa$.
3. *If* $\Delta \vdash \mathsf{M}_1 \longleftrightarrow \mathsf{M}_2 : \tau$ *then* $\Delta \vdash \mathsf{M}_1$ **is** M_2 **in** $[\![\tau]\!]$.
4. *If* $\Delta \vdash \mathsf{A}_1 \longleftrightarrow \mathsf{A}_2 : \kappa$ *then* $\Delta \vdash \mathsf{A}_1$ **is** A_2 **in** $[\![\kappa]\!]$.

Proof

By simultaneous structural induction on the simple types and kinds in the premises. We will show parts 1 and 3, to do with terms. The argument for type families is similar.

Case 1: Part (1), $\tau = \alpha$.

By definition of relation, $\Delta \vdash \mathsf{M}_1 \Longleftrightarrow \mathsf{M}_2 : \alpha$.

Case 2: Part (1), $\tau = \tau_1 \to \tau_2$.

$\Delta \vdash \mathsf{M}_1$ **is** M_2 **in** $[\![\tau_1 \to \tau_2]\!]$	By assumption
$\Delta, \mathsf{x}{:}\tau_1 \vdash \mathsf{x} \longleftrightarrow \mathsf{x} : \tau_1$	By rule (variable)
$\Delta, \mathsf{x}{:}\tau_1 \vdash \mathsf{x}$ **is** x **in** $[\![\tau_1]\!]$	By induction (part 3)
$\Delta, \mathsf{x}{:}\tau_1 \vdash \mathsf{M}_1\,\mathsf{x}$ **is** $\mathsf{M}_2\,\mathsf{x}$ **in** $[\![\tau_2]\!]$	By definition of relation
$\Delta, \mathsf{x}{:}\tau_1 \vdash \mathsf{M}_1\,\mathsf{x} \Longleftrightarrow \mathsf{M}_2\,\mathsf{x} : \tau_2$	By induction (part 1) on smaller type τ_2
$\Delta \vdash \mathsf{M}_1 \Longleftrightarrow \mathsf{M}_2 : \tau_1 \to \tau_2$	By rule (functions)

Case 3: Part (1), $\tau = \tau_1 \times \tau_2$.

$\Delta \vdash \mathsf{M}_1$ **is** M_2 **in** $[\![\tau_1 \times \tau_2]\!]$	By assumption
$\Delta \vdash \pi_1\mathsf{M}_1$ **is** $\pi_1\mathsf{M}_2$ **in** $[\![\tau_1]\!]$,	
$\Delta \vdash \pi_2\mathsf{M}_1$ **is** $\pi_2\mathsf{M}_2$ **in** $[\![\tau_2]\!]$	By definition of relation
$\Delta \vdash \pi_1\mathsf{M}_1 \Longleftrightarrow \pi_1\mathsf{M}_2 : \tau_1$,	
$\Delta \vdash \pi_2\mathsf{M}_1 \Longleftrightarrow \pi_2\mathsf{M}_2 : \tau_2$	By induction (part 1)
$\Delta \vdash \mathsf{M}_1 \Longleftrightarrow \mathsf{M}_2 : \tau_1 \times \tau_2$	By rule (products)

Case 4: Part (1), $\tau = 1^-$.

$\Delta \vdash \mathsf{M}_1 \Longleftrightarrow \mathsf{M}_2 : 1^-$ By rule (unit)

Case 5: Part (3), $\tau = \alpha$.

$\Delta \vdash \mathsf{M}_1 \longleftrightarrow \mathsf{M}_2 : \alpha$	By assumption
$\Delta \vdash \mathsf{M}_1 \Longleftrightarrow \mathsf{M}_2 : \alpha$	By rule (constants)
$\Delta \vdash \mathsf{M}_1$ **is** M_2 **in** $[\![\alpha]\!]$	By definition of relation

Case 6: Part (3), $\tau = \tau_1 \to \tau_2$.

$\Delta \vdash \mathsf{M}_1 \longleftrightarrow \mathsf{M}_2 : \tau_1 \to \tau_2$	By assumption
$\Delta_1 \vdash \mathsf{M}_{11}$ **is** M_{21} **in** $[\![\tau_1]\!]$ for an arbitrary $\Delta_1 \supseteq \Delta$	New hypothesis
$\Delta_1 \vdash \mathsf{M}_{11} \Longleftrightarrow \mathsf{M}_{21} : \tau_1$	By induction (part 1)
$\Delta_1 \vdash \mathsf{M}_1 \longleftrightarrow \mathsf{M}_2 : \tau_1 \to \tau_2$	By Weakening
$\Delta_1 \vdash \mathsf{M}_1\,\mathsf{M}_{11} \longleftrightarrow \mathsf{M}_2\,\mathsf{M}_{21} : \tau_2$	By rule (application)
$\Delta_1 \vdash \mathsf{M}_1\,\mathsf{M}_{11}$ **is** $\mathsf{M}_2\,\mathsf{M}_{21}$ **in** $[\![\tau_2]\!]$	By induction (part 3) on smaller type τ_2
$\Delta \vdash \mathsf{M}_1$ **is** M_2 **in** $[\![\tau_1 \to \tau_2]\!]$	By definition of rule

Case 7: Part (3), $\tau = \tau_1 \times \tau_2$.

$\Delta \vdash \mathsf{M}_1 \longleftrightarrow \mathsf{M}_2 : \tau_1 \times \tau_2$	By assumption
$\Delta \vdash \pi_i \mathsf{M}_1 \longleftrightarrow \pi_i \mathsf{M}_2 : \tau_i$	By rule (projection)
$\Delta \vdash \pi_i \mathsf{M}_1$ **is** $\pi_i \mathsf{M}_2$ **in** $[\![\tau_i]\!]$	By induction (part 3)
$\Delta \vdash \mathsf{M}_1$ **is** M_2 **in** $[\![\tau_1 \times \tau_2]\!]$	By definition of relation

Case 8: Part (3), $\tau = 1^-$.

$\Delta \vdash \mathsf{M}_1$ **is** M_2 **in** $[\![1^-]\!]$ By definition of relation

□

Similar to the simpler logical relations argument for proving injectivity, we will need closure under head expansion for proving completeness for the abstraction case. We will also need the symmetry and transitivity of the logical relations, which will use the corresponding properties at base types and lift to function and product types.

Lemma 7.2 (Closure Under Head Expansion)

1. *If* $\mathsf{M}_{11} \xrightarrow{\text{whr}} \mathsf{M}_{12}$ *and* $\Delta \vdash \mathsf{M}_{12}$ **is** M_2 **in** $[\![\tau]\!]$ *then* $\Delta \vdash \mathsf{M}_{11}$ **is** M_2 **in** $[\![\tau]\!]$.
2. *If* $\mathsf{A}_{11} \xrightarrow{\text{whr}} \mathsf{A}_{12}$ *and* $\Delta \vdash \mathsf{A}_{12}$ **is** A_2 **in** $[\![\kappa]\!]$ *then* $\Delta \vdash \mathsf{A}_{11}$ **is** A_2 **in** $[\![\kappa]\!]$.
3. *If* $\mathsf{M}_{21} \xrightarrow{\text{whr}} \mathsf{M}_{22}$ *and* $\Delta \vdash \mathsf{M}_1$ **is** M_{22} **in** $[\![\tau]\!]$ *then* $\Delta \vdash \mathsf{M}_1$ **is** M_{21} **in** $[\![\tau]\!]$.
4. *If* $\mathsf{A}_{21} \xrightarrow{\text{whr}} \mathsf{A}_{22}$ *and* $\Delta \vdash \mathsf{A}_1$ **is** A_{22} **in** $[\![\kappa]\!]$ *then* $\Delta \vdash \mathsf{A}_1$ **is** A_{21} **in** $[\![\kappa]\!]$.

Proof

By induction on the simple type or kind in the premises.

□

Lemma 7.3 (Weakening of Logical Relations)

1. *If* $\Delta \vdash \mathsf{M}_1$ **is** M_2 **in** $[\![\tau]\!]$ *and* $\Delta \subseteq \Delta^+$*, then* $\Delta^+ \vdash \mathsf{M}_1$ **is** M_2 **in** $[\![\tau]\!]$.
2. *If* $\Delta \vdash \mathsf{A}_1$ **is** A_2 **in** $[\![\kappa]\!]$ *and* $\Delta \subseteq \Delta^+$*, then* $\Delta^+ \vdash \mathsf{A}_1$ **is** A_2 **in** $[\![\kappa]\!]$.
3. *If* $\Delta \vdash \sigma_1$ **is** σ_2 **in** $[\![\Delta_1]\!]$ *and* $\Delta \subseteq \Delta^+$*, then* $\Delta^+ \vdash \sigma_1$ **is** σ_2 **in** $[\![\Delta_1]\!]$.

Proof

By induction on the simple type, kind or context in the premise.

□

Lemma 7.4 (Symmetry of Logical Relations)

1. *If* $\Delta \vdash \mathsf{M}_1$ **is** M_2 **in** $[\![\tau]\!]$ *then* $\Delta \vdash \mathsf{M}_2$ **is** M_1 **in** $[\![\tau]\!]$.
2. *If* $\Delta \vdash \mathsf{A}_1$ **is** A_2 **in** $[\![\kappa]\!]$ *then* $\Delta \vdash \mathsf{A}_2$ **is** A_1 **in** $[\![\kappa]\!]$.
3. *If* $\Delta \vdash \sigma_1$ **is** σ_2 **in** $[\![\Delta_1]\!]$ *then* $\Delta \vdash \sigma_2$ **is** σ_1 **in** $[\![\Delta_1]\!]$.

Proof

By induction on the simple type, kind or context in the premise. We will show the cases for objects. The other cases are similar.

Case 1: $\tau = \alpha$

$\Delta \vdash \mathsf{M}_1$ **is** M_2 **in** $[\![\alpha]\!]$	By assumption
$\Delta \vdash \mathsf{M}_1 \Longleftrightarrow \mathsf{M}_2 : \alpha$	By definition of relation
$\Delta \vdash \mathsf{M}_2 \Longleftrightarrow \mathsf{M}_1 : \alpha$	By Symmetry of Algorithm
$\Delta \vdash \mathsf{M}_2$ **is** M_1 **in** $[\![\alpha]\!]$	By definition of relation

Case 2: $\tau = \tau_1 \to \tau_2$

$\Delta \vdash \mathsf{M}_1$ **is** M_2 **in** $[\![\tau_1 \to \tau_2]\!]$	By assumption
$\Delta^+ \vdash \mathsf{M}_{21}$ **is** M_{11} **in** $[\![\tau_1]\!]$ for $\Delta^+ \supseteq \Delta$	New hypothesis
$\Delta^+ \vdash \mathsf{M}_{11}$ **is** M_{21} **in** $[\![\tau_1]\!]$	By induction
$\Delta^+ \vdash \mathsf{M}_1\, \mathsf{M}_{11}$ **is** $\mathsf{M}_2\, \mathsf{M}_{21}$ **in** $[\![\tau_2]\!]$	By definition of relation
$\Delta^+ \vdash \mathsf{M}_2\, \mathsf{M}_{21}$ **is** $\mathsf{M}_1\, \mathsf{M}_{11}$ **in** $[\![\tau_2]\!]$	By induction
$\Delta \vdash \mathsf{M}_2$ **is** M_1 **in** $[\![\tau_1 \to \tau_2]\!]$	By definition of relation

Case 3: $\tau = \tau_1 \times \tau_2$

$\Delta \vdash \mathsf{M}_1$ **is** M_2 **in** $[\![\tau_1 \times \tau_2]\!]$	By assumption
$\Delta \vdash \pi_i \mathsf{M}_1$ **is** $\pi_i \mathsf{M}_2$ **in** $[\![\tau_i]\!]$	By definition of relation
$\Delta \vdash \pi_i \mathsf{M}_2$ **is** $\pi_i \mathsf{M}_1$ **in** $[\![\tau_i]\!]$	By induction
$\Delta \vdash \mathsf{M}_2$ **is** M_1 **in** $[\![\tau_1 \to \tau_2]\!]$	By definition of relation

Case 4: $\tau = 1^-$

$\Delta \vdash \mathsf{M}_2$ **is** M_1 **in** $[\![1^-]\!]$ By definition of relation

□

Lemma 7.5 (Transitivity of Logical Relations)

1. *If* $\Delta \vdash \mathsf{M}_1$ **is** M_2 **in** $[\![\tau]\!]$ *and* $\Delta \vdash \mathsf{M}_2$ **is** M_3 **in** $[\![\tau]\!]$*, then* $\Delta \vdash \mathsf{M}_1$ **is** M_3 **in** $[\![\tau]\!]$*.*
2. *If* $\Delta \vdash \mathsf{A}_1$ **is** A_2 **in** $[\![\kappa]\!]$ *and* $\Delta \vdash \mathsf{A}_2$ **is** A_3 **in** $[\![\kappa]\!]$*, then* $\Delta \vdash \mathsf{A}_1$ **is** A_3 **in** $[\![\kappa]\!]$*.*
3. *If* $\Delta \vdash \sigma_1$ **is** σ_2 **in** $[\![\Delta_1]\!]$ *and* $\Delta \vdash \sigma_2$ **is** σ_3 **in** $[\![\Delta_1]\!]$*, then* $\Delta \vdash \sigma_1$ **is** σ_3 **in** $[\![\Delta_1]\!]$*.*

Proof

By induction on the simple type, kind or context in the premise. We will show the cases for objects. The other cases are similar.

Case 1: $\tau = \alpha$

$\Delta \vdash \mathsf{M}_1$ **is** M_2 **in** $[\![\alpha]\!]$ By assumption
$\Delta \vdash \mathsf{M}_2$ **is** M_3 **in** $[\![\alpha]\!]$ By assumption
$\Delta \vdash \mathsf{M}_1 \Longleftrightarrow \mathsf{M}_2 : \alpha$ By definition of relation
$\Delta \vdash \mathsf{M}_2 \Longleftrightarrow \mathsf{M}_3 : \alpha$ By definition of relation
$\Delta \vdash \mathsf{M}_1 \Longleftrightarrow \mathsf{M}_3 : \alpha$ By Transitivity of Algorithm
$\Delta \vdash \mathsf{M}_1$ **is** M_3 **in** $[\![\alpha]\!]$ By definition of relation

Case 2: $\tau = \tau_1 \to \tau_2$

$\Delta \vdash \mathsf{M}_1$ **is** M_2 **in** $[\![\tau_1 \to \tau_2]\!]$ By assumption
$\Delta \vdash \mathsf{M}_2$ **is** M_3 **in** $[\![\tau_1 \to \tau_2]\!]$ By assumption
$\Delta^+ \vdash \mathsf{M}_{11}$ **is** M_{21} **in** $[\![\tau_1]\!]$ for $\Delta^+ \supseteq \Delta$ New hypothesis
$\Delta^+ \vdash \mathsf{M}_{21}$ **is** M_{11} **in** $[\![\tau_1]\!]$ By Symmetry of Relation
$\Delta^+ \vdash \mathsf{M}_{21}$ **is** M_{21} **in** $[\![\tau_1]\!]$ By induction
$\Delta^+ \vdash \mathsf{M}_1\,\mathsf{M}_{11}$ **is** $\mathsf{M}_2\,\mathsf{M}_{21}$ **in** $[\![\tau_2]\!]$ By definition of relation
$\Delta^+ \vdash \mathsf{M}_2\,\mathsf{M}_{21}$ **is** $\mathsf{M}_3\,\mathsf{M}_{21}$ **in** $[\![\tau_2]\!]$ By definition of relation
$\Delta^+ \vdash \mathsf{M}_1\,\mathsf{M}_{11}$ **is** $\mathsf{M}_3\,\mathsf{M}_{21}$ **in** $[\![\tau_2]\!]$ By induction
$\Delta \vdash \mathsf{M}_1$ **is** M_3 **in** $[\![\tau_1 \to \tau_2]\!]$ By definition of relation

Case 3: $\tau = \tau_1 \times \tau_2$

$\Delta \vdash \mathsf{M}_1$ **is** M_2 **in** $[\![\tau_1 \times \tau_2]\!]$ By assumption
$\Delta \vdash \mathsf{M}_2$ **is** M_3 **in** $[\![\tau_1 \times \tau_2]\!]$ By assumption
$\Delta \vdash \pi_i\mathsf{M}_1$ **is** $\pi_i\mathsf{M}_2$ **in** $[\![\tau_i]\!]$ By definition of relation
$\Delta \vdash \pi_i\mathsf{M}_2$ **is** $\pi_i\mathsf{M}_3$ **in** $[\![\tau_i]\!]$ By definition of relation
$\Delta \vdash \pi_i\mathsf{M}_1$ **is** $\pi_i\mathsf{M}_3$ **in** $[\![\tau_i]\!]$ By induction
$\Delta \vdash \mathsf{M}_1$ **is** M_3 **in** $[\![\tau_1 \to \tau_2]\!]$ By definition of relation

Case 4: $\tau = 1^-$

$\Delta \vdash \mathsf{M}_1$ **is** M_3 **in** $[\![1^-]\!]$ By definition of relation

□

We now come to the main theorem about logical relations. This will enable us to directly get completeness of algorithmic equality.

Lemma 7.6 (Definitionally Equal Terms are Logically Related Under Substitutions)

1. *If* $\Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : \mathsf{A}$ *and* $\Delta \vdash \sigma_1$ **is** σ_2 **in** $[\![\Gamma^-]\!]$ *then* $\Delta \vdash \mathsf{M}_1[\sigma_1]$ **is** $\mathsf{M}_2[\sigma_2]$ **in** $[\![\mathsf{A}^-]\!]$.
2. *If* $\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_2 : \mathsf{K}$ *and* $\Delta \vdash \sigma_1$ **is** σ_2 **in** $[\![\Gamma^-]\!]$ *then* $\Delta \vdash \mathsf{A}_1[\sigma_1]$ **is** $\mathsf{A}_2[\sigma_2]$ **in** $[\![\mathsf{K}^-]\!]$.

Proof

By induction on the judgment of definitional equality. We will show some representative cases.

Case 1: $\dfrac{\Gamma(\mathsf{x}) = \mathsf{A}}{\Gamma \vdash \mathsf{x} = \mathsf{x} : \mathsf{A}}$

$\Delta \vdash \sigma_1$ **is** σ_2 **in** $[\![\Gamma^-]\!]$		By assumption
$\Delta \vdash \mathsf{M}_1$ **is** M_2 **in** $[\![\mathsf{A}^-]\!]$	for $[\mathsf{M}_1/\mathsf{x}] \in \sigma_1,\ [\mathsf{M}_2/\mathsf{x}] \in \sigma_2$	By definition of relation
$\Delta \vdash \mathsf{x}[\sigma_1]$ **is** $\mathsf{x}[\sigma_2]$ **in** $[\![\mathsf{A}^-]\!]$		By definition of substitution

Case 2: $\dfrac{\Sigma(\mathsf{c}) = \mathsf{A}}{\Gamma \vdash \mathsf{c} = \mathsf{c} : \mathsf{A}}$

$\Delta \vdash \mathsf{c} \longleftrightarrow \mathsf{c} : \mathsf{A}^-$	By rule (constant)
$\Delta \vdash \mathsf{c}$ **is** c **in** $[\![\mathsf{A}^-]\!]$	By Fundamental Theorem
$\Delta \vdash \mathsf{c}[\sigma_1]$ **is** $\mathsf{c}[\sigma_2]$ **in** $[\![\mathsf{A}^-]\!]$	By definition of substitution

Case 3: $\dfrac{\Gamma \vdash \mathsf{M}_{11} = \mathsf{M}_{21} : \Pi\mathsf{x}{:}\mathsf{A}_2.\mathsf{A}_1 \qquad \Gamma \vdash \mathsf{M}_{12} = \mathsf{M}_{22} : \mathsf{A}_2}{\Gamma \vdash \mathsf{M}_{11}\,\mathsf{M}_{12} = \mathsf{M}_{21}\,\mathsf{M}_{22} : \mathsf{A}_1\,[\mathsf{M}_{12}/\mathsf{x}]}$

$\Delta \vdash \mathsf{M}_{11}[\sigma_1]$ **is** $\mathsf{M}_{21}[\sigma_2]$ **in** $[\![\mathsf{A}_2{}^- \to \mathsf{A}_1{}^-]\!]$	By induction
$\Delta \vdash \mathsf{M}_{12}[\sigma_1]$ **is** $\mathsf{M}_{22}[\sigma_2]$ **in** $[\![\mathsf{A}_2{}^-]\!]$	By induction
$\Delta \vdash (\mathsf{M}_{11}[\sigma_1])\,(\mathsf{M}_{12}[\sigma_2])$ **is** $(\mathsf{M}_{21}[\sigma_2])\,(\mathsf{M}_{22}[\sigma_2])$ **in** $[\![\mathsf{A}_1{}^-]\!]$	By definition of relation
$\Delta \vdash (\mathsf{M}_{11}\,\mathsf{M}_{12}[\sigma_1])$ **is** $(\mathsf{M}_{21}\,\mathsf{M}_{22}[\sigma_2])$ **in** $[\![\mathsf{A}_1{}^-]\!]$	By definition of substitution

Case 4: $\dfrac{\Gamma \vdash \mathsf{A}_{11} = \mathsf{A}_1 : \mathsf{type} \qquad \Gamma \vdash \mathsf{A}_{12} = \mathsf{A}_1 : \mathsf{type} \qquad \Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{M}_1 = \mathsf{M}_2 : \mathsf{A}_2}{\Gamma \vdash \lambda\mathsf{x}{:}\mathsf{A}_{11}.\mathsf{M}_1 = \lambda\mathsf{x}{:}\mathsf{A}_{12}.\mathsf{M}_2 : \Pi\mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2}$

$\Delta^+ \vdash \mathsf{M}_{11}$ **is** M_{21} **in** $[\![\mathsf{A}_1{}^-]\!]$	for $\Delta^+ \supseteq \Delta$	New hypothesis
$\Delta^+ \vdash \sigma_1$ **is** σ_2 **in** $[\![\Gamma^-]\!]$		By weakening of relation
$\Delta^+ \vdash \sigma_1, \mathsf{M}_{11}/\mathsf{x}$ **is** $\sigma_2, \mathsf{M}_{21}/\mathsf{x}$ **in** $[\![\Gamma^-, \mathsf{x}{:}\mathsf{A}_1{}^-]\!]$		By definition of relation
$\Delta^+ \vdash \mathsf{M}_1[\sigma_1, \mathsf{M}_{11}/\mathsf{x}]$ **is** $\mathsf{M}_2[\sigma_2, \mathsf{M}_{21}/\mathsf{x}]$ **in** $[\![\mathsf{A}_2{}^-]\!]$		By induction
$\Delta^+ \vdash (\lambda\mathsf{x}{:}\mathsf{A}_{11}.\mathsf{M}_1[\sigma_1, \mathsf{x}/\mathsf{x}])\,\mathsf{M}_{11}$ **is** $\mathsf{M}_2[\sigma_2, \mathsf{M}_{21}/\mathsf{x}]$ **in** $[\![\mathsf{A}_2{}^-]\!]$		By Closure Under Head Expansion
$\Delta^+ \vdash (\lambda\mathsf{x}{:}\mathsf{A}_{11}.\mathsf{M}_1[\sigma_1, \mathsf{x}/\mathsf{x}])\,\mathsf{M}_{11}$ **is** $(\lambda\mathsf{x}{:}\mathsf{A}_{12}.\mathsf{M}_2[\sigma_2, \mathsf{x}/\mathsf{x}])\,\mathsf{M}_{21}$ **in** $[\![\mathsf{A}_2{}^-]\!]$		By Closure Under Head Expansion
$\Delta^+ \vdash ((\lambda\mathsf{x}{:}\mathsf{A}_{11}.\mathsf{M}_1)[\sigma_1])\,\mathsf{M}_{11}$ **is** $((\lambda\mathsf{x}{:}\mathsf{A}_{12}.\mathsf{M}_2)[\sigma_2])\,\mathsf{M}_{21}$ **in** $[\![\mathsf{A}_2{}^-]\!]$		By definition of substitution
$\Delta \vdash (\lambda\mathsf{x}{:}\mathsf{A}_{11}.\mathsf{M}_1)[\sigma_1]$ **is** $(\lambda\mathsf{x}{:}\mathsf{A}_{12}.\mathsf{M}_2)[\sigma_2]$ **in** $[\![\mathsf{A}_1{}^- \to \mathsf{A}_2{}^-]\!]$		By definition of relation

Case 5: $$\frac{\Gamma \vdash \mathsf{A}_1 : \mathsf{type} \quad \Gamma \vdash \mathsf{M}_1 : \Pi\mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 \quad \Gamma \vdash \mathsf{M}_2 : \Pi\mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 \quad \Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{M}_1\,\mathsf{x} = \mathsf{M}_2\,\mathsf{x} : \mathsf{A}_2}{\Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : \Pi\mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2}$$

$\Delta^+ \vdash \mathsf{M}_{11}$ **is** M_{21} **in** $[\![\mathsf{A}_1{}^-]\!]$ for $\Delta^+ \supseteq \Delta$ — New hypothesis
$\Delta^+ \vdash \sigma_1$ **is** σ_2 **in** $[\![\Gamma^-]\!]$ — By weakening of relation
$\Delta^+ \vdash \sigma_1, \mathsf{M}_{11}/\mathsf{x}$ **is** $\sigma_2, \mathsf{M}_{21}/\mathsf{x}$ **in** $[\![\Gamma^-, \mathsf{x}{:}\mathsf{A}_1{}^-]\!]$ — By definition of relation
$\Delta^+ \vdash (\mathsf{M}_1\,\mathsf{x})[\sigma_1, \mathsf{M}_{11}/\mathsf{x}]$ **is** $(\mathsf{M}_2\,\mathsf{x})[\sigma_2, \mathsf{M}_{21}/\mathsf{x}]$ **in** $[\![\mathsf{A}_2{}^-]\!]$ — By induction
$\Delta^+ \vdash (\mathsf{M}_1[\sigma_1])\,\mathsf{M}_{11}$ **is** $(\mathsf{M}_2[\sigma_2])\,\mathsf{M}_{21}$ **in** $[\![\mathsf{A}_2{}^-]\!]$ — By definition of substitution
$\Delta \vdash \mathsf{M}_1[\sigma_1]$ **is** $\mathsf{M}_2[\sigma_2]$ **in** $[\![\mathsf{A}_1{}^- \to \mathsf{A}_2{}^-]\!]$ — By definition of relation

Case 6: $$\frac{\Gamma \vdash \mathsf{A}_1 : \mathsf{type} \qquad \Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{M}_{12} = \mathsf{M}_{22} : \mathsf{A}_2 \qquad \Gamma \vdash \mathsf{M}_{11} = \mathsf{M}_{21} : \mathsf{A}_1}{\Gamma \vdash (\lambda\mathsf{x}{:}\mathsf{A}_1.\mathsf{M}_{12})\,\mathsf{M}_{11} = \mathsf{M}_{22}\,[\mathsf{M}_{21}/\mathsf{x}] : \mathsf{A}_2\,[\mathsf{M}_{11}/\mathsf{x}]}$$

$\Delta \vdash \sigma_1$ **is** σ_2 **in** $[\![\Gamma^-]\!]$ — By assumption
$\Delta \vdash \mathsf{M}_{11}$ **is** M_{21} **in** $[\![\mathsf{A}_1{}^-]\!]$ — By induction
$\Delta \vdash \sigma_1, (\mathsf{M}_{11}[\sigma_1])/\mathsf{x}$ **is** $\sigma_2, (\mathsf{M}_{21}[\sigma_2])/\mathsf{x}$ **in** $[\![\Gamma^-, \mathsf{x}{:}\mathsf{A}_1{}^-]\!]$ — By definition of relation
$\Delta \vdash \mathsf{M}_{12}[\sigma_1, (\mathsf{M}_{11}[\sigma_1])/\mathsf{x}]$ **is** $\mathsf{M}_{22}[\sigma_2, (\mathsf{M}_{21}[\sigma_2])/\mathsf{x}]$ **in** $[\![\mathsf{A}_2{}^-]\!]$ — By induction
$\Delta \vdash (\mathsf{M}_{12}[\sigma_1, \mathsf{x}/\mathsf{x}])[(\mathsf{M}_{11}[\sigma_1])/\mathsf{x}]$ **is** $\mathsf{M}_{22}[\sigma_2, (\mathsf{M}_{21}[\sigma_2])/\mathsf{x}]$ **in** $[\![\mathsf{A}_2{}^-]\!]$ — By definition of substitution
$\Delta \vdash (\mathsf{M}_{12}[\sigma_1, \mathsf{x}/\mathsf{x}])[(\mathsf{M}_{11}[\sigma_1])/\mathsf{x}]$ **is** $(\mathsf{M}_{22}[\mathsf{M}_{21}/\mathsf{x}])[\sigma_2]$ **in** $[\![\mathsf{A}_2{}^-]\!]$ — By definition of substitution
$\Delta \vdash (\lambda\mathsf{x}{:}\mathsf{A}_1.\mathsf{M}_{12}[\sigma_1, \mathsf{x}/\mathsf{x}])\,\mathsf{M}_{11}[\sigma_1]$ **is** $(\mathsf{M}_{22}[\mathsf{M}_{21}/\mathsf{x}])[\sigma_2]$ **in** $[\![\mathsf{A}_2{}^-]\!]$ — By Closure Under Head Expansion
$\Delta \vdash (\lambda\mathsf{x}{:}\mathsf{A}_1.\mathsf{M}_{12}\,\mathsf{M}_{11})[\sigma_1]$ **is** $(\mathsf{M}_{22}[\mathsf{M}_{21}/\mathsf{x}])[\sigma_2]$ **in** $[\![\mathsf{A}_2{}^-]\!]$ — By definition of substitution
$\Delta \vdash (\lambda\mathsf{x}{:}\mathsf{A}_1.\mathsf{M}_{12}\,\mathsf{M}_{11})[\sigma_1]$ **is** $(\mathsf{M}_{22}[\mathsf{M}_{21}/\mathsf{x}])[\sigma_2]$ **in** $[\![\mathsf{A}_2\,[\mathsf{M}_{11}/\mathsf{x}]^-]\!]$ — By Erasure Preservation

Case 7: $$\frac{\Gamma \vdash \Sigma\mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 : \mathsf{type} \qquad \Gamma \vdash \mathsf{M}_{11} = \mathsf{M}_{21} : \mathsf{A}_1 \qquad \Gamma \vdash \mathsf{M}_{12} = \mathsf{M}_{22} : \mathsf{A}_2\,[\mathsf{M}_{11}/\mathsf{x}]}{\Gamma \vdash \langle \mathsf{M}_{11}, \mathsf{M}_{12}\rangle^{\Sigma\mathsf{x}:\mathsf{A}_1.\mathsf{A}_2} = \langle \mathsf{M}_{21}, \mathsf{M}_{22}\rangle^{\Sigma\mathsf{x}:\mathsf{A}_1.\mathsf{A}_2} : \Sigma\mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2}$$

$\Delta \vdash \mathsf{M}_{11}[\sigma_1]$ **is** $\mathsf{M}_{21}[\sigma_2]$ **in** $[\![\mathsf{A}_1{}^-]\!]$ — By induction
$\Delta \vdash \pi_1\langle(\mathsf{M}_{11}[\sigma_1]), (\mathsf{M}_{12}[\sigma_1])\rangle^{(\Sigma\mathsf{x}:\mathsf{A}_1.\mathsf{A}_2[\sigma_1])}$ **is** $\mathsf{M}_{21}[\sigma_2]$ **in** $[\![\mathsf{A}_1{}^-]\!]$ — By Closure Under Head Expansion
$\Delta \vdash \pi_1\langle(\mathsf{M}_{11}[\sigma_1]), (\mathsf{M}_{12}[\sigma_1])\rangle^{(\Sigma\mathsf{x}:\mathsf{A}_1.\mathsf{A}_2[\sigma_1])}$ **is** $\pi_1\langle(\mathsf{M}_{21}[\sigma_2]), (\mathsf{M}_{22}[\sigma_2])\rangle^{(\Sigma\mathsf{x}:\mathsf{A}_1.\mathsf{A}_2[\sigma_2])}$ **in** $[\![\mathsf{A}_1{}^-]\!]$
By Closure Under Head Expansion
$\Delta \vdash \pi_1(\langle \mathsf{M}_{11}, \mathsf{M}_{12}\rangle^{\Sigma\mathsf{x}:\mathsf{A}_1.\mathsf{A}_2}[\sigma_1])$ **is** $\pi_1(\langle \mathsf{M}_{21}, \mathsf{M}_{22}\rangle^{\Sigma\mathsf{x}:\mathsf{A}_1.\mathsf{A}_2}[\sigma_2])$ **in** $[\![\mathsf{A}_1{}^-]\!]$ — By definition of substitution
$\Delta \vdash \mathsf{M}_{12}[\sigma_1]$ **is** $\mathsf{M}_{22}[\sigma_2]$ **in** $[\![\mathsf{A}_2{}^-]\!]$ — By induction
$\Delta \vdash \pi_2\langle(\mathsf{M}_{11}[\sigma_1]), (\mathsf{M}_{12}[\sigma_1])\rangle^{(\Sigma\mathsf{x}:\mathsf{A}_1.\mathsf{A}_2[\sigma_1])}$ **is** $\mathsf{M}_{22}[\sigma_2]$ **in** $[\![\mathsf{A}_2{}^-]\!]$ — By Closure Under Head Expansion
$\Delta \vdash \pi_2\langle(\mathsf{M}_{11}[\sigma_1]), (\mathsf{M}_{12}[\sigma_1])\rangle^{(\Sigma\mathsf{x}:\mathsf{A}_1.\mathsf{A}_2[\sigma_1])}$ **is** $\pi_2\langle(\mathsf{M}_{21}[\sigma_2]), (\mathsf{M}_{22}[\sigma_2])\rangle^{(\Sigma\mathsf{x}:\mathsf{A}_1.\mathsf{A}_2[\sigma_2])}$ **in** $[\![\mathsf{A}_2{}^-]\!]$
By Closure Under Head Expansion
$\Delta \vdash \pi_2(\langle \mathsf{M}_{11}, \mathsf{M}_{12}\rangle^{\Sigma\mathsf{x}:\mathsf{A}_1.\mathsf{A}_2}[\sigma_1])$ **is** $\pi_2(\langle \mathsf{M}_{21}, \mathsf{M}_{22}\rangle^{\Sigma\mathsf{x}:\mathsf{A}_1.\mathsf{A}_2}[\sigma_2])$ **in** $[\![\mathsf{A}_2{}^-]\!]$ — By definition of substitution
$\Delta \vdash (\langle \mathsf{M}_{11}, \mathsf{M}_{12}\rangle^{\Sigma\mathsf{x}:\mathsf{A}_1.\mathsf{A}_2})[\sigma_1]$ **is** $(\langle \mathsf{M}_{21}, \mathsf{M}_{22}\rangle^{\Sigma\mathsf{x}:\mathsf{A}_1.\mathsf{A}_2})[\sigma_2]$ **in** $[\![\mathsf{A}_1{}^- \times \mathsf{A}_2{}^-]\!]$ — By definition of relation

Case 8: $$\frac{\Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : \Sigma\mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2}{\Gamma \vdash \pi_1\mathsf{M}_1 = \pi_1\mathsf{M}_2 : \mathsf{A}_1}$$

$\Delta \vdash \mathsf{M}_1[\sigma_1]$ **is** $\mathsf{M}_2[\sigma_2]$ **in** $[\![\mathsf{A}_1{}^- \times \mathsf{A}_2{}^-]\!]$ — By induction
$\Delta \vdash \pi_1(\mathsf{M}_1[\sigma_1])$ **is** $\pi_1(\mathsf{M}_2[\sigma_2])$ **in** $[\![\mathsf{A}_1{}^-]\!]$ — By definition of relation
$\Delta \vdash (\pi_1\mathsf{M}_1)[\sigma_1]$ **is** $(\pi_1\mathsf{M}_2)[\sigma_2]$ **in** $[\![\mathsf{A}_1{}^-]\!]$ — By definition of substitution

Case 9: $$\frac{\Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2}{\Gamma \vdash \pi_2 \mathsf{M}_1 = \pi_2 \mathsf{M}_2 : \mathsf{A}_2\,[\pi_1 \mathsf{M}_1/\mathsf{x}]}$$

$\Delta \vdash \mathsf{M}_1[\sigma_1]$ **is** $\mathsf{M}_2[\sigma_2]$ **in** $[\![\mathsf{A}_1{}^- \times \mathsf{A}_2{}^-]\!]$	By induction
$\Delta \vdash \pi_2(\mathsf{M}_1[\sigma_1])$ **is** $\pi_2(\mathsf{M}_2[\sigma_2])$ **in** $[\![\mathsf{A}_2{}^-]\!]$	By definition of relation
$\Delta \vdash (\pi_2\mathsf{M}_1)[\sigma_1]$ **is** $(\pi_2\mathsf{M}_2)[\sigma_2]$ **in** $[\![\mathsf{A}_2{}^-]\!]$	By definition of substitution

Case 10: $$\frac{\Gamma \vdash \mathsf{M}_1 : 1 \quad \Gamma \vdash \mathsf{M}_2 : 1}{\Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : 1}$$

$\Delta \vdash \mathsf{M}_1[\sigma_1]$ **is** $\mathsf{M}_2[\sigma_2]$ **in** $[\![1^-]\!]$	By definition of relation

Case 11: $$\frac{\Gamma \vdash \mathsf{M}_1 = \mathsf{M}_3 : \mathsf{A}_1 \qquad \Gamma \vdash \mathsf{M}_2 : \mathsf{A}_2}{\Gamma \vdash \pi_1 \langle \mathsf{M}_1, \mathsf{M}_2 \rangle^{\mathsf{A}} = \mathsf{M}_3 : \mathsf{A}_1}$$

$\Delta \vdash \mathsf{M}_1[\sigma_1]$ **is** $\mathsf{M}_3[\sigma_2]$ **in** $[\![\mathsf{A}_1{}^-]\!]$	By induction
$\Delta \vdash \pi_1\langle(\mathsf{M}_1[\sigma_1]), (\mathsf{M}_2[\sigma_1])\rangle^{(\mathsf{A}[\sigma_1])}$ **is** $\mathsf{M}_3[\sigma_2]$ **in** $[\![\mathsf{A}_1{}^-]\!]$	By Closure Under Head Expansion
$\Delta \vdash (\pi_1\langle \mathsf{M}_1, \mathsf{M}_2\rangle^{\mathsf{A}})[\sigma_1]$ **is** $\mathsf{M}_3[\sigma_2]$ **in** $[\![\mathsf{A}_1{}^-]\!]$	By definition of substitution

Case 12: $$\frac{\Gamma \vdash \mathsf{M}_1 : \mathsf{A}_1 \qquad \Gamma \vdash \mathsf{M}_2 = \mathsf{M}_3 : \mathsf{A}_2}{\Gamma \vdash \pi_2 \langle \mathsf{M}_1, \mathsf{M}_2 \rangle^{\mathsf{A}} = \mathsf{M}_3 : \mathsf{A}_2}$$

$\Delta \vdash \mathsf{M}_2[\sigma_1]$ **is** $\mathsf{M}_3[\sigma_2]$ **in** $[\![\mathsf{A}_2{}^-]\!]$	By induction
$\Delta \vdash \pi_2\langle(\mathsf{M}_1[\sigma_1]), (\mathsf{M}_2[\sigma_1])\rangle^{(\mathsf{A}[\sigma_1])}$ **is** $\mathsf{M}_3[\sigma_2]$ **in** $[\![\mathsf{A}_2{}^-]\!]$	By Closure Under Head Expansion
$\Delta \vdash (\pi_2\langle \mathsf{M}_1, \mathsf{M}_2\rangle^{\mathsf{A}})[\sigma_1]$ **is** $\mathsf{M}_3[\sigma_2]$ **in** $[\![\mathsf{A}_2{}^-]\!]$	By definition of substitution

Case 13:
$$\frac{\Gamma \vdash \mathsf{M}_1 : \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 \quad \Gamma \vdash \mathsf{M}_2 : \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 \quad \Gamma \vdash \pi_1\mathsf{M}_1 = \pi_1\mathsf{M}_2 : \mathsf{A}_1 \quad \Gamma \vdash \pi_2\mathsf{M}_1 = \pi_2\mathsf{M}_2 : \mathsf{A}_2\,[\pi_1\mathsf{M}_1/\mathsf{x}]}{\Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2}$$

$\Delta \vdash (\pi_1\mathsf{M}_1)[\sigma_1]$ **is** $(\pi_1\mathsf{M}_2)[\sigma_2]$ **in** $[\![\mathsf{A}_1{}^-]\!]$	By induction
$\Delta \vdash (\pi_2\mathsf{M}_1)[\sigma_1]$ **is** $(\pi_2\mathsf{M}_2)[\sigma_2]$ **in** $[\![\mathsf{A}_2{}^-]\!]$	By induction
$\Delta \vdash \pi_1(\mathsf{M}_1[\sigma_1])$ **is** $\pi_1(\mathsf{M}_2[\sigma_2])$ **in** $[\![\mathsf{A}_1{}^-]\!]$	By definition of substitution
$\Delta \vdash \pi_2(\mathsf{M}_1[\sigma_1])$ **is** $\pi_2(\mathsf{M}_2[\sigma_2])$ **in** $[\![\mathsf{A}_2{}^-]\!]$	By definition of substitution
$\Delta \vdash \mathsf{M}_1[\sigma_1]$ **is** $\mathsf{M}_2[\sigma_2]$ **in** $[\![\mathsf{A}_1{}^- \times \mathsf{A}_2{}^-]\!]$	By definition of relation

Case 14: $$\frac{\Gamma \vdash \mathsf{M}_2 = \mathsf{M}_1 : \mathsf{A}}{\Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : \mathsf{A}}$$

$\Delta \vdash \sigma_1$ **is** σ_2 **in** $[\![\Gamma^-]\!]$	By assumption
$\Delta \vdash \sigma_2$ **is** σ_1 **in** $[\![\Gamma^-]\!]$	By Symmetry of Relation
$\Delta \vdash \mathsf{M}_2[\sigma_2]$ **is** $\mathsf{M}_1[\sigma_1]$ **in** $[\![\mathsf{A}^-]\!]$	By induction
$\Delta \vdash \mathsf{M}_1[\sigma_1]$ **is** $\mathsf{M}_2[\sigma_2]$ **in** $[\![\mathsf{A}^-]\!]$	By Symmetry of Relation

Case 15: $\dfrac{\Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : \mathsf{A} \qquad \Gamma \vdash \mathsf{M}_2 = \mathsf{M}_3 : \mathsf{A}}{\Gamma \vdash \mathsf{M}_1 = \mathsf{M}_3 : \mathsf{A}}$

$\Delta \vdash \sigma_1$ **is** σ_2 **in** $[\![\Gamma^-]\!]$	By assumption
$\Delta \vdash \sigma_2$ **is** σ_1 **in** $[\![\Gamma^-]\!]$	By Symmetry of Relation
$\Delta \vdash \sigma_2$ **is** σ_2 **in** $[\![\Gamma^-]\!]$	By Transitivity of Relation
$\Delta \vdash \mathsf{M}_1[\sigma_1]$ **is** $\mathsf{M}_2[\sigma_2]$ **in** $[\![\mathsf{A}^-]\!]$	By induction
$\Delta \vdash \mathsf{M}_2[\sigma_2]$ **is** $\mathsf{M}_3[\sigma_2]$ **in** $[\![\mathsf{A}^-]\!]$	By induction
$\Delta \vdash \mathsf{M}_1[\sigma_1]$ **is** $\mathsf{M}_3[\sigma_2]$ **in** $[\![\mathsf{A}^-]\!]$	By Transitivity of Relation

Case 16: $\dfrac{\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_2 : \mathsf{type} \qquad \Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : \mathsf{A}_2}{\Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : \mathsf{A}_1}$

$\mathsf{A}_1{}^- = \mathsf{A}_2{}^-$	By Erasure Preservation
$\Delta \vdash \mathsf{M}_1[\sigma_1]$ **is** $\mathsf{M}_2[\sigma_2]$ **in** $[\![\mathsf{A}_2{}^-]\!]$	By induction
$\Delta \vdash \mathsf{M}_1[\sigma_1]$ **is** $\mathsf{M}_2[\sigma_2]$ **in** $[\![\mathsf{A}_1{}^-]\!]$	From previous

Case 17: $\dfrac{\Gamma \vdash \mathsf{A}_{11} = \mathsf{A}_{21} : \mathsf{type} \qquad \Gamma \vdash \mathsf{A}_{11} : \mathsf{type} \qquad \Gamma, \mathsf{x}{:}\mathsf{A}_{11} \vdash \mathsf{A}_{12} = \mathsf{A}_{22} : \mathsf{type}}{\Gamma \vdash \Sigma \mathsf{x}{:}\mathsf{A}_{11}.\mathsf{A}_{12} = \Sigma \mathsf{x}{:}\mathsf{A}_{21}.\mathsf{A}_{22} : \mathsf{type}}$

$\Delta \vdash \mathsf{A}_{11}[\sigma_1]$ **is** $\mathsf{A}_{21}[\sigma_2]$ **in** $[\![\mathsf{type}^-]\!]$	By induction
$\Delta \vdash \mathsf{A}_{11}[\sigma_1] \Longleftrightarrow \mathsf{A}_{21}[\sigma_2] : \mathsf{type}^-$	By definition of relation
$\Delta, \mathsf{x}{:}\mathsf{A}_{11}{}^- \vdash \mathsf{x} \longleftrightarrow \mathsf{x} : \mathsf{A}_{11}{}^-$	By rule (variable)
$\Delta, \mathsf{x}{:}\mathsf{A}_{11}{}^- \vdash \mathsf{x}$ **is** x **in** $[\![\mathsf{A}_{11}{}^-]\!]$	By Fundamental Theorem
$\Delta, \mathsf{x}{:}\mathsf{A}_{11}{}^- \vdash \sigma_1\,[\mathsf{x}/\mathsf{x}]$ **is** $\sigma_2\,[\mathsf{x}/\mathsf{x}]$ **in** $[\![\mathsf{A}_{11}{}^-]\!]$	By definition of relation
$\Delta, \mathsf{x}{:}\mathsf{A}_{11}{}^- \vdash \mathsf{A}_{12}[\sigma_1, \mathsf{x}/\mathsf{x}]$ **is** $\mathsf{A}_{22}[\sigma_2, \mathsf{x}/\mathsf{x}]$ **in** $[\![\mathsf{type}^-]\!]$	By induction
$\Delta, \mathsf{x}{:}\mathsf{A}_{11}{}^- \vdash \mathsf{A}_{12}[\sigma_1, \mathsf{x}/\mathsf{x}] \Longleftrightarrow \mathsf{A}_{22}[\sigma_2, \mathsf{x}/\mathsf{x}] : \mathsf{type}^-$	By definition of relation
$\Delta \vdash \Sigma \mathsf{x}{:}\mathsf{A}_{11}[\sigma_1].\mathsf{A}_{12}[\sigma_1, \mathsf{x}/\mathsf{x}] \longleftrightarrow \Sigma \mathsf{x}{:}\mathsf{A}_{21}[\sigma_2].\mathsf{A}_{22}[\sigma_2, \mathsf{x}/\mathsf{x}] : \mathsf{type}^-$	By rule (sums)
$\Delta \vdash \Sigma \mathsf{x}{:}\mathsf{A}_{11}[\sigma_1].\mathsf{A}_{12}[\sigma_1, \mathsf{x}/\mathsf{x}]$ **is** $\Sigma \mathsf{x}{:}\mathsf{A}_{21}[\sigma_2].\mathsf{A}_{22}[\sigma_2, \mathsf{x}/\mathsf{x}]$ **in** $[\![\mathsf{type}^-]\!]$	By definition of relation
$\Delta \vdash (\Sigma \mathsf{x}{:}\mathsf{A}_{11}.\mathsf{A}_{12})[\sigma_1]$ **is** $(\Sigma \mathsf{x}{:}\mathsf{A}_{21}.\mathsf{A}_{22})[\sigma_2]$ **in** $[\![\mathsf{type}^-]\!]$	By definition of substitution

Case 18: $\dfrac{}{\Gamma \vdash 1 = 1 : \mathsf{type}}$

$\Delta \vdash 1 \longleftrightarrow 1 : \mathsf{type}^-$	By rule (unit)
$\Delta \vdash 1$ **is** 1 **in** $[\![\mathsf{type}^-]\!]$	By definition of relation
$\Delta \vdash 1[\sigma_1]$ **is** $1[\sigma_2]$ **in** $[\![\mathsf{type}^-]\!]$	By definition of substitution

□

We are almost done with proving completeness. We first need an easy lemma to show that identity substitutions are logically related to themselves.

Lemma 7.7 (Identity Substitutions are Logically Related) $\Gamma^- \vdash \mathsf{id}_\Gamma$ **is** id_Γ **in** $[\![\Gamma^-]\!]$.

Proof

By induction on the structure of Γ.

□

Theorem 7.8 (Definitionally Equal Terms are Logically Related)

1. *If* $\Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : \mathsf{A}$ *then* $\Gamma^- \vdash \mathsf{M}_1$ **is** M_2 **in** $[\![\mathsf{A}^-]\!]$.

2. *If* $\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_2 : \mathsf{K}$ *then* $\Gamma^- \vdash \mathsf{A}_1$ **is** A_2 **in** $[\![\mathsf{K}^-]\!]$.

Proof

Direct, by Lemma 7.6 and Lemma 7.7. □

Theorem 7.9 (Completeness of Equality Algorithm)

1. *If* $\Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : \mathsf{A}$ *then* $\Gamma^- \vdash \mathsf{M}_1 \Longleftrightarrow \mathsf{M}_2 : \mathsf{A}^-$.

2. *If* $\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_2 : \mathsf{K}$ *then* $\Gamma^- \vdash \mathsf{A}_1 \Longleftrightarrow \mathsf{A}_2 : \mathsf{K}^-$.

Proof

Direct, by Lemma 7.8 and Lemma 7.1. □

8 Soundness of the Algorithm and Canonical Forms

We will need to prove a few simple lemmas before we prove our algorithm sound. The first one says that the weak-head reduction relation is sound. This is important because at base types the algorithm performs weak-head reductions.

Lemma 8.1 (Subject Reduction)

1. *If* $\mathsf{M}_1 \xrightarrow{\text{whr}} \mathsf{M}_2$ *and* $\Gamma \vdash \mathsf{M}_1 : \mathsf{A}$ *then* $\Gamma \vdash \mathsf{M}_2 : \mathsf{A}$ *and* $\Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : \mathsf{A}$.

2. *If* $\mathsf{A}_1 \xrightarrow{\text{whr}} \mathsf{A}_2$ *and* $\Gamma \vdash \mathsf{A}_1 : \mathsf{K}$ *then* $\Gamma \vdash \mathsf{A}_2 : \mathsf{K}$ *and* $\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_2 : \mathsf{K}$.

Proof

By induction on the definition of weak head reduction.

□

Lemma 8.2 (Inversion on Erasure)

1. *If* $\Gamma \vdash \mathsf{A} : \mathsf{type}$ *and* $\mathsf{A}^- = 1^-$ *then* $\mathsf{A} \xrightarrow{\text{whr}}{}^{*}\ 1$ *and* $\Gamma \vdash \mathsf{A} = 1 : \mathsf{type}$.

2. *If* $\Gamma \vdash \mathsf{A} : \mathsf{type}$ *and* $\mathsf{A}^- = \tau_1 \to \tau_2$ *then* $\mathsf{A} \xrightarrow{\text{whr}}{}^{*}\ \Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2$ *and* $\Gamma \vdash \mathsf{A} = \Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 : \mathsf{type}$.

3. *If* $\Gamma \vdash \mathsf{A} : \mathsf{type}$ *and* $\mathsf{A}^- = \tau_1 \times \tau_2$ *then* $\mathsf{A} \xrightarrow{\text{whr}}{}^{*}\ \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2$ *and* $\Gamma \vdash \mathsf{A} = \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 : \mathsf{type}$.

Proof

By induction on the length of a weak head normal reduction starting from A.

Case 1: $\mathsf{A} \stackrel{\text{whr}}{\not\longrightarrow}$

By Erasure Corresponds to Weak Head Normal Terms and Subject Reduction

Case 2: $\mathsf{A} \xrightarrow{\text{whr}} \mathsf{A}'$

By induction and transitivity

□

8.1 Canonical Forms

We will actually prove soundness and canonical forms properties together for our language. We now define canonical and atomic objects, families and kinds, which are a syntactic subset of the corresponding terms, defined by the following grammar.

Canonical Kinds	$\bar{\mathsf{K}} ::=$	type	kind of types
	$\mid$	$\Pi \mathsf{x}{:}\bar{\bar{\mathsf{A}}}.\bar{\mathsf{K}}$	dependent product kind
Atomic Families	$\bar{\mathsf{A}} ::=$	a	family constants
	$\mid$	$\bar{\mathsf{A}}\,\bar{\bar{\mathsf{M}}}$	family application
	$\mid$	$\Pi \mathsf{x}{:}\bar{\bar{\mathsf{A}}}_1.\bar{\bar{\mathsf{A}}}_2$	family of functions
	$\mid$	$\Sigma \mathsf{x}{:}\bar{\bar{\mathsf{A}}}_1.\bar{\bar{\mathsf{A}}}_2$	family of products
	$\mid$	1	unit type
Canonical Families	$\bar{\bar{\mathsf{A}}} ::=$	$\bar{\mathsf{A}}$	atomic families
	$\mid$	$\lambda \mathsf{x}{:}\mathsf{A}_1.\bar{\bar{\mathsf{A}}}_2$	family level abstraction
Atomic Objects	$\bar{\mathsf{M}} ::=$	c	object constants
	$\mid$	x	object variables
	$\mid$	$\bar{\mathsf{M}}_1\,\bar{\bar{\mathsf{M}}}_2$	object level application
	$\mid$	$\pi_i\bar{\mathsf{M}} \quad (i = 1, 2)$	projections from pairs
Canonical Objects	$\bar{\bar{\mathsf{M}}} ::=$	$\bar{\mathsf{M}}$	atomic objects
	$\mid$	$\langle \bar{\bar{\mathsf{M}}}_1, \bar{\bar{\mathsf{M}}}_2 \rangle^{\mathsf{A}}$	pairs of objects
	$\mid$	$\langle\rangle$	unit object
	$\mid$	$\lambda \mathsf{x}{:}\mathsf{A}.\bar{\bar{\mathsf{M}}}$	object functions

Notice that these are different from the original notion of canonical forms. A better term might be quasi-canonical forms, or almost canonical forms, but that term is already used in Harper and Pfenning for something else. The difference from the original canonical forms is that type annotations of abstractions at both term and family levels need not be in canonical form. This is the same in spirit to the quasi-canonical form of Harper and Pfenning [5], but those forms elide type annotations on abstractions, so that the canonical forms do not belong syntactically to the language of LF.

We can now prove our main lemma, which implies both soundness of the algorithm for equality as well as existence of canonical forms.

Lemma 8.3 (Algorithm produces Canonical Mediating Terms)

1. *If* $\Gamma \vdash \mathsf{M}_1 : \mathsf{A}$, $\Gamma \vdash \mathsf{M}_2 : \mathsf{A}$ *and* $\Gamma^- \vdash \mathsf{M}_1 \Longleftrightarrow \mathsf{M}_2 : \mathsf{A}^-$, *then there is a canonical object* $\bar{\bar{\mathsf{M}}}$ *such that* $\Gamma \vdash \bar{\bar{\mathsf{M}}} : \mathsf{A}$, $\Gamma \vdash \mathsf{M}_1 = \bar{\bar{\mathsf{M}}} : \mathsf{A}$ *and* $\Gamma \vdash \mathsf{M}_2 = \bar{\bar{\mathsf{M}}} : \mathsf{A}$.

2. *If* $\Gamma \vdash \mathsf{M}_1 : \mathsf{A}_1$, $\Gamma \vdash \mathsf{M}_2 : \mathsf{A}_2$ *and* $\Gamma^- \vdash \mathsf{M}_1 \longleftrightarrow \mathsf{M}_2 : \tau$, *then there is an atomic object* $\bar{\mathsf{M}}$ *such that* $\Gamma \vdash \bar{\mathsf{M}} : \mathsf{A}_1$, $\Gamma \vdash \mathsf{M}_1 = \bar{\mathsf{M}} : \mathsf{A}_1$ *and* $\Gamma \vdash \mathsf{M}_2 = \bar{\mathsf{M}} : \mathsf{A}_1$ *and* $\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_2 : \mathsf{type}$ *and* ${\mathsf{A}_1}^- = {\mathsf{A}_2}^- = \tau$.

3. *If* $\Gamma \vdash \mathsf{A}_1 : \mathsf{K}$, $\Gamma \vdash \mathsf{A}_2 : \mathsf{K}$ *and* $\Gamma^- \vdash \mathsf{A}_1 \Longleftrightarrow \mathsf{A}_2 : \mathsf{K}^-$, *then there exists a canonical family* $\bar{\bar{\mathsf{A}}}$ *such that* $\Gamma \vdash \bar{\bar{\mathsf{A}}} : \mathsf{K}$, $\Gamma \vdash \mathsf{A}_1 = \bar{\bar{\mathsf{A}}} : \mathsf{K}$ *and* $\Gamma \vdash \mathsf{A}_2 = \bar{\bar{\mathsf{A}}} : \mathsf{K}$.

4. *If* $\Gamma \vdash \mathsf{A}_1 : \mathsf{K}_1$, $\Gamma \vdash \mathsf{A}_2 : \mathsf{K}_2$ *and* $\Gamma^- \vdash \mathsf{A}_1 \longleftrightarrow \mathsf{A}_2 : \kappa$, *then there exists an atomic family* $\bar{\mathsf{A}}$ *such that* $\Gamma \vdash \bar{\mathsf{A}} : \mathsf{K}_1$, $\Gamma \vdash \mathsf{A}_1 = \bar{\mathsf{A}} : \mathsf{K}_1$ *and* $\Gamma \vdash \mathsf{A}_2 = \bar{\mathsf{A}} : \mathsf{K}_1$ *and* $\Gamma \vdash \mathsf{K}_1 = \mathsf{K}_2 : \mathsf{kind}$ *and* ${\mathsf{K}_1}^- = {\mathsf{K}_2}^- = \kappa$.

5. *If* $\Gamma \vdash \mathsf{K}_1 : \mathsf{kind}$, $\Gamma \vdash \mathsf{K}_2 : \mathsf{kind}$ *and* $\Gamma^- \vdash \mathsf{K}_1 \longleftrightarrow \mathsf{K}_2 : \mathsf{kind}^-$, *then there exists a canonical kind* $\bar{\mathsf{K}}$ *such that* $\Gamma \vdash \bar{\mathsf{K}} : \mathsf{kind}$, $\Gamma \vdash \mathsf{K}_1 = \bar{\mathsf{K}} : \mathsf{kind}$ *and* $\Gamma \vdash \mathsf{K}_2 = \bar{\mathsf{K}} : \mathsf{kind}$.

Proof

By induction on the algorithmic judgment. We will show some representative cases.

Case 1:
$$\frac{\mathsf{M}_1 \xrightarrow{\text{whr}} \mathsf{M}_2 \quad \Delta \vdash \mathsf{M}_2 \Longleftrightarrow \mathsf{M} : \alpha}{\Delta \vdash \mathsf{M}_1 \Longleftrightarrow \mathsf{M} : \alpha}$$

$\Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : \mathsf{A}$ By Subject Reduction
$\Gamma \vdash \mathsf{M}_2 : \mathsf{A}$ By Regularity
There exists a canonical object $\bar{\bar{\mathsf{M}}}$ such that
$\Gamma \vdash \bar{\bar{\mathsf{M}}} : \mathsf{A}$,
$\Gamma \vdash \mathsf{M}_2 = \bar{\bar{\mathsf{M}}} : \mathsf{A}$,
$\Gamma \vdash \mathsf{M} = \bar{\bar{\mathsf{M}}} : \mathsf{A}$ By induction
$\Gamma \vdash \mathsf{M}_1 = \bar{\bar{\mathsf{M}}} : \mathsf{A}$ By rule (transitivity)

Case 2: $$\frac{\Delta \vdash \mathsf{M}_1 \longleftrightarrow \mathsf{M}_2 : \alpha}{\Delta \vdash \mathsf{M}_1 \Longleftrightarrow \mathsf{M}_2 : \alpha}$$

There exists an atomic term $\bar{\mathsf{M}}$ such that
$\Gamma \vdash \bar{\mathsf{M}} : \mathsf{A}$,
$\Gamma \vdash \mathsf{M}_1 = \bar{\mathsf{M}} : \mathsf{A}$,
$\Gamma \vdash \mathsf{M}_2 = \bar{\mathsf{M}} : \mathsf{A}$ By induction

Case 3: $$\frac{\Delta, \mathsf{x}{:}\tau_1 \vdash \mathsf{M}_1\, \mathsf{x} \Longleftrightarrow \mathsf{M}_2\, \mathsf{x} : \tau_2}{\Delta \vdash \mathsf{M}_1 \Longleftrightarrow \mathsf{M}_2 : \tau_1 \to \tau_2}$$

$\Gamma \vdash \mathsf{A} = \Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 : \mathsf{type}$,
${\mathsf{A}_1}^- = \tau_1$ and ${\mathsf{A}_2}^- = \tau_2$ By Inversion on Erasure
$\Gamma \vdash \Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 : \mathsf{type}$ By Regularity
$\Gamma \vdash \mathsf{A}_1 : \mathsf{type}$ By Inversion on Typing
$\Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{A}_2 : \mathsf{type}$ By Inversion on Typing
$\Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{M}_1\, \mathsf{x} : \mathsf{A}_2$ By rule (application typing)
$\Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{M}_2\, \mathsf{x} : \mathsf{A}_2$ By rule (application typing)
There exists a canonical object $\bar{\bar{\mathsf{M}}}$ such that
$\Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \bar{\bar{\mathsf{M}}} : \mathsf{A}_2$,
$\Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{M}_1\, \mathsf{x} = \bar{\bar{\mathsf{M}}} : \mathsf{A}_2$ By induction
$\Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{M}_2\, \mathsf{x} = \bar{\bar{\mathsf{M}}} : \mathsf{A}_2$ By induction
$\Gamma \vdash \lambda \mathsf{x}{:}\mathsf{A}_1.\bar{\bar{\mathsf{M}}} : \Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2$ By rule
$\Gamma \vdash \lambda \mathsf{x}{:}\mathsf{A}_1.\bar{\bar{\mathsf{M}}} : \mathsf{A}$ By rule (type conversion)
$\Gamma \vdash \mathsf{M}_1 = \lambda \mathsf{x}{:}\mathsf{A}_1.\bar{\bar{\mathsf{M}}} : \Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2$ By rule (extensionality)
$\Gamma \vdash \mathsf{M}_2 = \lambda \mathsf{x}{:}\mathsf{A}_1.\bar{\bar{\mathsf{M}}} : \Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2$ By rule (extensionality)
$\Gamma \vdash \mathsf{M}_1 = \lambda \mathsf{x}{:}\mathsf{A}_1.\bar{\bar{\mathsf{M}}} : \mathsf{A}$ By rule (type conversion)
$\Gamma \vdash \mathsf{M}_2 = \lambda \mathsf{x}{:}\mathsf{A}_1.\bar{\bar{\mathsf{M}}} : \mathsf{A}$ By rule (type conversion)

Case 4: $$\frac{\Delta \vdash \pi_1 \mathsf{M}_1 \Longleftrightarrow \pi_1 \mathsf{M}_2 : \tau_1 \quad \Delta \vdash \pi_2 \mathsf{M}_1 \Longleftrightarrow \pi_2 \mathsf{M}_2 : \tau_2}{\Delta \vdash \mathsf{M}_1 \Longleftrightarrow \mathsf{M}_2 : \tau_1 \times \tau_2}$$

$\Gamma \vdash \mathsf{A} = \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 : \mathsf{type}$,
${\mathsf{A}_1}^- = \tau_1$ and ${\mathsf{A}_2}^- = \tau_2$ By Inversion on Erasure
$\Gamma \vdash \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 : \mathsf{type}$ By Regularity
$\Gamma \vdash \mathsf{A}_1 : \mathsf{type}$ By Inversion on Typing
$\Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{A}_2 : \mathsf{type}$ By Inversion on Typing
$\Gamma \vdash \mathsf{M}_1 : \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2$ By assumption
$\Gamma \vdash \mathsf{M}_2 : \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2$ By assumption

$\Gamma \vdash \pi_1 M_1 : A_1$	By rule (projection)
$\Gamma \vdash \pi_1 M_2 : A_1$	By rule (projection)
$\Gamma \vdash \pi_2 M_1 : A_2\,[\pi_1 M_1/x]$	By rule (projection)
$\Gamma \vdash \pi_2 M_2 : A_2\,[\pi_1 M_2/x]$	By rule (projection)
There exists a canonical object $\bar{\bar{M}}_1$ such that	
$\Gamma \vdash \bar{\bar{M}}_1 : A_1$,	
$\Gamma \vdash \pi_1 M_1 = \bar{\bar{M}}_1 : A_1$,	
$\Gamma \vdash \pi_1 M_2 = \bar{\bar{M}}_1 : A_1$	By induction
$\Gamma \vdash A_2\,[\pi_1 M_2/x] = A_2\,[\pi_1 M_1/x] : \mathsf{type}$	By Functionality
$\Gamma \vdash \pi_2 M_2 : A_2\,[\pi_1 M_1/x]$	By rule (type conversion)
There exists a canonical object $\bar{\bar{M}}_2$ such that	
$\Gamma \vdash \bar{\bar{M}}_2 : A_2\,[\pi_1 M_1/x]$,	
$\Gamma \vdash \pi_2 M_1 = \bar{\bar{M}}_2 : A_2\,[\pi_1 M_1/x]$,	
$\Gamma \vdash \pi_2 M_2 = \bar{\bar{M}}_2 : A_2\,[\pi_1 M_1/x]$	By induction
$\Gamma \vdash \langle \bar{\bar{M}}_1, \bar{\bar{M}}_2 \rangle^{\Sigma x:A_1.A_2} : \Sigma x:A_1.A_2$	By rule
$\Gamma \vdash \langle \bar{\bar{M}}_1, \bar{\bar{M}}_2 \rangle^{\Sigma x:A_1.A_2} : A$	By rule (type conversion)
$\Gamma \vdash M_1 = \langle \bar{\bar{M}}_1, \bar{\bar{M}}_2 \rangle^{\Sigma x:A_1.A_2} : \Sigma x:A_1.A_2$	By rule (extensionality)
$\Gamma \vdash M_2 = \langle \bar{\bar{M}}_1, \bar{\bar{M}}_2 \rangle^{\Sigma x:A_1.A_2} : \Sigma x:A_1.A_2$	By rule (extensionality)
$\Gamma \vdash M_1 = \langle \bar{\bar{M}}_1, \bar{\bar{M}}_2 \rangle^{\Sigma x:A_1.A_2} : A$	By rule (type conversion)
$\Gamma \vdash M_2 = \langle \bar{\bar{M}}_1, \bar{\bar{M}}_2 \rangle^{\Sigma x:A_1.A_2} : A$	By rule (type conversion)

Case 5: $$\frac{}{\Delta \vdash M_1 \Longleftrightarrow M_2 : 1}$$

$\Gamma \vdash A = 1 : \mathsf{type}$	By Inversion on Erasure
$\Gamma \vdash M_1 : 1$	By rule (type conversion)
$\Gamma \vdash M_2 : 1$	By rule (type conversion)
$\Gamma \vdash \langle\rangle : 1$	By rule
$\Gamma \vdash \langle\rangle : A$	By rule (type conversion)
$\Gamma \vdash M_1 = \langle\rangle : 1$	By rule
$\Gamma \vdash M_2 = \langle\rangle : 1$	By rule
$\Gamma \vdash M_1 = \langle\rangle : A$	By rule (type conversion)
$\Gamma \vdash M_2 = \langle\rangle : A$	By rule (type conversion)

Case 6: $$\frac{\Delta(x) = \tau}{\Delta \vdash x \longleftrightarrow x : \tau}$$

$\Gamma \vdash x : A_1$	By assumption
$\Gamma \vdash x : A_2$	By assumption
$\Gamma(x) = A$, $\Gamma \vdash A = A_1 : \mathsf{type}$, $\Gamma \vdash A = A_2 : \mathsf{type}$	By Inversion
$\Gamma \vdash x = x : A$	By rule (variable equality)
$\Gamma \vdash x = x : A_1$	By rule (type conversion)
$\Gamma \vdash A_1 = A_2 : \mathsf{type}$	By rules (symmetry, transitivity)
${A_1}^- = {A_2}^- = A^- = \tau$	By Erasure Preservation

Case 7: $$\frac{\Delta \vdash M_{11} \longleftrightarrow M_{21} : \tau_2 \to \tau_1 \quad \Delta \vdash M_{12} \Longleftrightarrow M_{22} : \tau_2}{\Delta \vdash M_{11}\, M_{12} \longleftrightarrow M_{21}\, M_{22} : \tau_1}$$

$\Gamma \vdash M_{11}\, M_{12} : A_1$	By assumption
$\Gamma \vdash M_{21}\, M_{22} : A_2$	By assumption
$\Gamma \vdash M_{11} : \Pi x{:}A_{11}.A_{12},$	
$\Gamma \vdash M_{12} : A_{11},$	
$\Gamma \vdash A_{12}\,[M_{12}/x] = A_1 : \mathsf{type}$	By typing inversion
$\Gamma \vdash M_{21} : \Pi x{:}A_{21}.A_{22},$	
$\Gamma \vdash M_{22} : A_{21},$	
$\Gamma \vdash A_{22}\,[M_{22}/x] = A_2 : \mathsf{type}$	By typing inversion
There is an atomic object $\bar{M}$ such that	
$\Gamma \vdash \bar{M} : \Pi x{:}A_{11}.A_{12},$	
$\Gamma \vdash M_{11} = \bar{M} : \Pi x{:}A_{11}.A_{12},$	
$\Gamma \vdash M_{21} = \bar{M} : \Pi x{:}A_{11}.A_{12},$	
$\Gamma \vdash \Pi x{:}A_{11}.A_{12} = \Pi x{:}A_{21}.A_{22} : \mathsf{type}$	
$(\Pi x{:}A_{11}.A_{12})^{-} = (\Pi x{:}A_{21}.A_{22})^{-} = \tau_2 \rightarrow \tau_1$	By induction
$\Gamma \vdash A_{11} = A_{21} : \mathsf{type},$	
$\Gamma, x{:}A_{11} \vdash A_{12} = A_{22} : \mathsf{type}$	By Injectivity of Products
${A_{11}}^{-} = {A_{21}}^{-} = \tau_2$	By definition of $()^{-}$
${A_{12}}^{-} = {A_{22}}^{-} = \tau_1$	By definition of $()^{-}$
$\Gamma \vdash M_{22} : A_{11}$	By rule (type conversion)
There exists a canonical object $\bar{\bar{M}}$ such that	
$\Gamma \vdash \bar{\bar{M}} : A_{11},$	
$\Gamma \vdash M_{12} = \bar{\bar{M}} : A_{11},$	
$\Gamma \vdash M_{22} = \bar{\bar{M}} : A_{11}$	By induction
$\Gamma \vdash A_{12}\left[\bar{\bar{M}}/x\right] = A_{12}\,[M_{12}/x] : \mathsf{type}$	By Functionality
$\Gamma \vdash \bar{M}\,\bar{\bar{M}} : A_{12}\left[\bar{\bar{M}}/x\right]$	By rule (application)
$\Gamma \vdash \bar{M}\,\bar{\bar{M}} : A_{12}\,[M_{12}/x]$	By rule (type conversion)
$\Gamma \vdash M_{11}\, M_{12} = \bar{M}\,\bar{\bar{M}} : A_{12}\,[M_{12}/x]$	By rule (application)
$\Gamma \vdash M_{11}\, M_{12} = \bar{M}\,\bar{\bar{M}} : A_1$	By rule (type conversion)
$\Gamma \vdash M_{21}\, M_{22} = \bar{M}\,\bar{\bar{M}} : A_{12}\,[M_{12}/x]$	By rule (application)
$\Gamma \vdash M_{21}\, M_{22} = \bar{M}\,\bar{\bar{M}} : A_1$	By rule (type conversion)
$\Gamma \vdash A_{12}\,[M_{12}/x] = A_{22}\,[M_{22}/x] : \mathsf{type}$	By Functionality
$\Gamma \vdash A_1 = A_2 : \mathsf{type}$	By rules (symmetry, transitivity)
${A_1}^{-} = {A_{12}}^{-} = {A_{22}}^{-} = {A_2}^{-} = \tau_1$	By Erasure Preservation

Case 8: $$\frac{\Delta \vdash M_1 \longleftrightarrow M_2 : \tau_1 \times \tau_2}{\Delta \vdash \pi_1 M_1 \longleftrightarrow \pi_1 M_2 : \tau_1}$$

$\Gamma \vdash \pi_1 M_1 : A_1$	By assumption
$\Gamma \vdash \pi_1 M_2 : A_2$	By assumption
$\Gamma \vdash M_1 : \Sigma x{:}A_1.A_{11}$	By Inversion
$\Gamma \vdash M_2 : \Sigma x{:}A_2.A_{21}$	By Inversion
There exists an atomic object $\bar{M}$ such that	
$\Gamma \vdash \bar{M} : \Sigma x{:}A_3.A_{31},$	
$\Gamma \vdash M_1 = \bar{M} : \Sigma x{:}A_3.A_{31},$	
$\Gamma \vdash M_2 = \bar{M} : \Sigma x{:}A_3.A_{31},$	
$\Gamma \vdash \Sigma x{:}A_1.A_{11} = \Sigma x{:}A_3.A_{31} : \mathsf{type},$	
$\Gamma \vdash \Sigma x{:}A_2.A_{21} = \Sigma x{:}A_3.A_{31} : \mathsf{type},$	
$\Sigma x{:}A_1.{A_{11}}^{-} = \Sigma x{:}A_2.{A_{21}}^{-} = \Sigma x{:}A_3.{A_{31}}^{-} = \tau_1 \times \tau_2$	By induction
$\Gamma \vdash \pi_1 \bar{M} : A_3$	By rule
$\Gamma \vdash \pi_1 M_1 = \pi_1 \bar{M} : A_3$	By rule (projection)

$\Gamma \vdash \pi_1 \mathsf{M}_2 = \pi_1 \bar{\mathsf{M}} : \mathsf{A}_3$ By rule (projection)
$\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_3 : \mathsf{type}$ By Injectivity
$\Gamma \vdash \pi_1 \mathsf{M}_1 = \pi_1 \bar{\mathsf{M}} : \mathsf{A}_1$ By rule (type conversion)
$\Gamma \vdash \pi_1 \mathsf{M}_2 = \pi_1 \bar{\mathsf{M}} : \mathsf{A}_1$ By rule (type conversion)
$\Gamma \vdash \mathsf{A}_2 = \mathsf{A}_3 : \mathsf{type}$ By Injectivity
$\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_2 : \mathsf{type}$ By rules (symmetry, transitivity)
${\mathsf{A}_1}^- = {\mathsf{A}_2}^- = {\mathsf{A}_3}^- = \tau_1$ By Erasure Preservation

Case 9: $$\frac{\Delta \vdash \mathsf{M}_1 \longleftrightarrow \mathsf{M}_2 : \tau_1 \times \tau_2}{\Delta \vdash \pi_2 \mathsf{M}_1 \longleftrightarrow \pi_2 \mathsf{M}_2 : \tau_2}$$

$\Gamma \vdash \pi_2 \mathsf{M}_1 : \mathsf{A}_1$ By assumption
$\Gamma \vdash \pi_2 \mathsf{M}_2 : \mathsf{A}_2$ By assumption
$\Gamma \vdash \mathsf{M}_1 : \Sigma \mathsf{x}{:}\mathsf{A}_{11}.\mathsf{A}_{12}$,
$\Gamma \vdash \mathsf{A}_{12}\,[\pi_1 \mathsf{M}_1/\mathsf{x}] = \mathsf{A}_1 : \mathsf{type}$ By Inversion
$\Gamma \vdash \mathsf{M}_2 : \Sigma \mathsf{x}{:}\mathsf{A}_{21}.\mathsf{A}_{22}$,
$\Gamma \vdash \mathsf{A}_{22}\,[\pi_1 \mathsf{M}_2/\mathsf{x}] = \mathsf{A}_2 : \mathsf{type}$ By Inversion
There exists an atomic object $\bar{\mathsf{M}}$ such that
$\Gamma \vdash \bar{\mathsf{M}} : \Sigma \mathsf{x}{:}\mathsf{A}_{31}.\mathsf{A}_{32}$,
$\Gamma \vdash \mathsf{M}_1 = \bar{\mathsf{M}} : \Sigma \mathsf{x}{:}\mathsf{A}_{31}.\mathsf{A}_{32}$,
$\Gamma \vdash \mathsf{M}_2 = \bar{\mathsf{M}} : \Sigma \mathsf{x}{:}\mathsf{A}_{31}.\mathsf{A}_{32}$,
$\Gamma \vdash \Sigma \mathsf{x}{:}\mathsf{A}_{11}.\mathsf{A}_{12} = \Sigma \mathsf{x}{:}\mathsf{A}_{31}.\mathsf{A}_{32} : \mathsf{type}$,
$\Gamma \vdash \Sigma \mathsf{x}{:}\mathsf{A}_{21}.\mathsf{A}_{22} = \Sigma \mathsf{x}{:}\mathsf{A}_{31}.\mathsf{A}_{32} : \mathsf{type}$,
$(\Sigma \mathsf{x}{:}\mathsf{A}_{11}.\mathsf{A}_{12})^- = (\Sigma \mathsf{x}{:}\mathsf{A}_{21}.\mathsf{A}_{22})^- = (\Sigma \mathsf{x}{:}\mathsf{A}_{31}.\mathsf{A}_{32})^- = \tau_1 \times \tau_2$ By induction
$\Gamma \vdash \pi_1 \mathsf{M}_1 = \pi_1 \bar{\mathsf{M}} : \mathsf{A}_{31}$ By rule (projection)
$\Gamma \vdash \pi_1 \mathsf{M}_2 = \pi_1 \bar{\mathsf{M}} : \mathsf{A}_{31}$ By rule (projection)
$\Gamma \vdash \pi_2 \mathsf{M}_1 = \pi_2 \bar{\mathsf{M}} : \mathsf{A}_{32}\,[\pi_1 \mathsf{M}_1/\mathsf{x}]$ By rule (projection)
$\Gamma \vdash \pi_2 \mathsf{M}_2 = \pi_2 \bar{\mathsf{M}} : \mathsf{A}_{32}\,[\pi_1 \mathsf{M}_2/\mathsf{x}]$ By rule (projection)
$\Gamma, \mathsf{x}{:}\mathsf{A}_{31} \vdash \mathsf{A}_{12} = \mathsf{A}_{32} : \mathsf{type}$,
$\Gamma, \mathsf{x}{:}\mathsf{A}_{31} \vdash \mathsf{A}_{22} = \mathsf{A}_{32} : \mathsf{type}$, By Injectivity
$\Gamma \vdash \mathsf{A}_{12}\,[\pi_1 \mathsf{M}_1/\mathsf{x}] = \mathsf{A}_{22}\,[\pi_1 \mathsf{M}_2/\mathsf{x}] : \mathsf{type}$ By Functionality
$\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_2 : \mathsf{type}$ By rules (symmetry, transitivity)
${\mathsf{A}_1}^- = {\mathsf{A}_2}^- = {\mathsf{A}_{32}}^- = \tau_2$ By Erasure Preservation

Case 10: $$\frac{\Delta \vdash \mathsf{A}_{11} \Longleftrightarrow \mathsf{A}_{21} : \mathsf{type}^- \quad \Delta, \mathsf{x}{:}{\mathsf{A}_{11}}^- \vdash \mathsf{A}_{12} \Longleftrightarrow \mathsf{A}_{22} : \mathsf{type}^-}{\Delta \vdash \Pi \mathsf{x}{:}\mathsf{A}_{11}.\mathsf{A}_{12} \longleftrightarrow \Pi \mathsf{x}{:}\mathsf{A}_{21}.\mathsf{A}_{22} : \mathsf{type}^-}$$

$\Gamma \vdash \Pi \mathsf{x}{:}\mathsf{A}_{11}.\mathsf{A}_{12} : \mathsf{K}_1$ By assumption
$\Gamma \vdash \Pi \mathsf{x}{:}\mathsf{A}_{21}.\mathsf{A}_{22} : \mathsf{K}_2$ By assumption
$\Gamma \vdash \mathsf{K}_1 = \mathsf{type} : \mathsf{kind}$,
$\Gamma \vdash \mathsf{K}_2 = \mathsf{type} : \mathsf{kind}$ By Inversion
$\Gamma \vdash \mathsf{A}_{11} : \mathsf{type}$,
$\Gamma \vdash \mathsf{A}_{21} : \mathsf{type}$ By Inversion
There is a canonical family $\bar{\bar{\mathsf{A}}}_1$ such that
$\Gamma \vdash \bar{\bar{\mathsf{A}}}_1 : \mathsf{type}$,
$\Gamma \vdash \mathsf{A}_{11} = \bar{\bar{\mathsf{A}}} : \mathsf{type}$,
$\Gamma \vdash \mathsf{A}_{21} = \bar{\bar{\mathsf{A}}} : \mathsf{type}$ By induction
$\Gamma, \mathsf{x}{:}\mathsf{A}_{11} \vdash \mathsf{A}_{12} : \mathsf{type}$,
$\Gamma, \mathsf{x}{:}\mathsf{A}_{21} \vdash \mathsf{A}_{22} : \mathsf{type}$ By inversion

$\Gamma, x{:}\bar{\bar{A}}_1 \vdash A_{22} : \mathsf{type}$ By Context Conversion
$\Gamma, x{:}\bar{\bar{A}}_1 \vdash A_{22} : \mathsf{type}$ By Context Conversion
There is a canonical family $\bar{\bar{A}}_2$ such that
$\Gamma, x{:}\bar{\bar{A}}_1 \vdash \bar{\bar{A}}_2 : \mathsf{type}$,
$\Gamma, x{:}\bar{\bar{A}}_1 \vdash A_{12} = \bar{\bar{A}}_2 : \mathsf{type}$ By induction
$\Gamma, x{:}\bar{\bar{A}}_1 \vdash A_{22} = \bar{\bar{A}}_2 : \mathsf{type}$ By induction
$\Gamma \vdash \Pi x{:}\bar{\bar{A}}_1.\bar{\bar{A}}_2 : \mathsf{type}$ By rule
$\Gamma \vdash \Pi x{:}A_{11}.A_{12} = \Pi x{:}\bar{\bar{A}}_1.\bar{\bar{A}}_2 : \mathsf{type}$ By rule (Product Equality)
$\Gamma \vdash \Pi x{:}A_{21}.A_{22} = \Pi x{:}\bar{\bar{A}}_1.\bar{\bar{A}}_2 : \mathsf{type}$ By rule (Product Equality)
$K_1{}^- = K_2{}^- = \mathsf{type}^-$ By Erasure Preservation

Case 11: $$\frac{\Delta \vdash A_{11} \Longleftrightarrow A_{21} : \mathsf{type}^- \quad \Delta, x{:}A_{11}{}^- \vdash A_{12} \Longleftrightarrow A_{22} : \mathsf{type}^-}{\Delta \vdash \Sigma x{:}A_{11}.A_{12} \longleftrightarrow \Sigma x{:}A_{21}.A_{22} : \mathsf{type}^-}$$

$\Gamma \vdash \Sigma x{:}A_{11}.A_{12} : K_1$ By assumption
$\Gamma \vdash \Sigma x{:}A_{21}.A_{22} : K_2$ By assumption
$\Gamma \vdash K_1 = \mathsf{type} : \mathsf{kind}$,
$\Gamma \vdash K_2 = \mathsf{type} : \mathsf{kind}$ By Inversion
$\Gamma \vdash A_{11} : \mathsf{type}$,
$\Gamma \vdash A_{21} : \mathsf{type}$ By Inversion
There is a canonical family $\bar{\bar{A}}_1$ such that
$\Gamma \vdash \bar{\bar{A}}_1 : \mathsf{type}$,
$\Gamma \vdash A_{11} = \bar{\bar{A}} : \mathsf{type}$,
$\Gamma \vdash A_{21} = \bar{\bar{A}} : \mathsf{type}$ By induction
$\Gamma, x{:}A_{11} \vdash A_{12} : \mathsf{type}$,
$\Gamma, x{:}A_{21} \vdash A_{22} : \mathsf{type}$ By inversion
$\Gamma, x{:}\bar{\bar{A}}_1 \vdash A_{22} : \mathsf{type}$ By Context Conversion
$\Gamma, x{:}\bar{\bar{A}}_1 \vdash A_{22} : \mathsf{type}$ By Context Conversion
There is a canonical family $\bar{\bar{A}}_2$ such that
$\Gamma, x{:}\bar{\bar{A}}_1 \vdash \bar{\bar{A}}_2 : \mathsf{type}$,
$\Gamma, x{:}\bar{\bar{A}}_1 \vdash A_{12} = \bar{\bar{A}}_2 : \mathsf{type}$ By induction
$\Gamma, x{:}\bar{\bar{A}}_1 \vdash A_{22} = \bar{\bar{A}}_2 : \mathsf{type}$ By induction
$\Gamma \vdash \Sigma x{:}\bar{\bar{A}}_1.\bar{\bar{A}}_2 : \mathsf{type}$ By rule
$\Gamma \vdash \Sigma x{:}A_{11}.A_{12} = \Sigma x{:}\bar{\bar{A}}_1.\bar{\bar{A}}_2 : \mathsf{type}$ By rule (Product Equality)
$\Gamma \vdash \Sigma x{:}A_{21}.A_{22} = \Sigma x{:}\bar{\bar{A}}_1.\bar{\bar{A}}_2 : \mathsf{type}$ By rule (Product Equality)
$K_1{}^- = K_2{}^- = \mathsf{type}^-$ By Erasure Preservation

□

Theorem 8.4 (Soundness)

1. *If* $\Gamma \vdash M_1 : A$, $\Gamma \vdash M_2 : A$ *and* $\Gamma^- \vdash M_1 \Longleftrightarrow M_2 : A^-$, *then* $\Gamma \vdash M_1 = M_2 : A$.
2. *If* $\Gamma \vdash M_1 : A_1$, $\Gamma \vdash M_2 : A_2$ *and* $\Gamma^- \vdash M_1 \longleftrightarrow M_2 : \tau$, *then* $\Gamma \vdash M_1 = M_2 : A_1$, $\Gamma \vdash A_1 = A_2 : \mathsf{type}$ *and* $A_1{}^- = A_2{}^- = \tau$.
3. *If* $\Gamma \vdash A_1 : K$, $\Gamma \vdash A_2 : K$ *and* $\Gamma^- \vdash A_1 \Longleftrightarrow A_2 : K^-$, *then* $\Gamma \vdash A_1 = A_2 : K$.
4. *If* $\Gamma \vdash A_1 : K_1$, $\Gamma \vdash A_2 : K_2$ *and* $\Gamma^- \vdash A_1 \longleftrightarrow A_2 : \kappa$, *then* $\Gamma \vdash A_1 = A_2 : K_1$, $\Gamma \vdash K_1 = K_2 : \mathsf{kind}$ *and* $K_1{}^- = K_2{}^- = \kappa$.
5. *If* $\Gamma \vdash K_1 : \mathsf{kind}$, $\Gamma \vdash K_2 : \mathsf{kind}$ *and* $\Gamma^- \vdash K_1 \longleftrightarrow K_2 : \mathsf{kind}^-$, *then* $\Gamma \vdash K_1 = K_2 : \mathsf{kind}$.

Proof

Directly, from previous lemma. We will show the cases for objects, the cases for families and kinds are similar.

Case 1:

$\Gamma \vdash \mathsf{M}_1 = \bar{\bar{\mathsf{M}}} : \mathsf{A}$,
$\Gamma \vdash \mathsf{M}_2 = \bar{\bar{\mathsf{M}}} : \mathsf{A}$ By Canonical Mediating Terms for Algorithm
$\Gamma \vdash \bar{\bar{\mathsf{M}}} = \mathsf{M}_2 : \mathsf{A}$ By rule (symmetry)
$\Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : \mathsf{A}$ By rule (transitivity)

Case 2:

$\Gamma \vdash \mathsf{M}_1 = \bar{\mathsf{M}} : \mathsf{A}_1$,
$\Gamma \vdash \mathsf{M}_2 = \bar{\mathsf{M}} : \mathsf{A}_1$,
$\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_2 : \mathsf{type}$,
${\mathsf{A}_1}^- = {\mathsf{A}_2}^- = \tau$ By Canonical Mediating Terms for Algorithm
$\Gamma \vdash \bar{\mathsf{M}} = \mathsf{M}_2 : \mathsf{A}_1$ By rule (symmetry)
$\Gamma \vdash \mathsf{M}_1 = \mathsf{M}_2 : \mathsf{A}_1$ By rule (transitivity)

□

We can now also prove canonical forms by appeal to lemma 8.3.

Theorem 8.5 (Canonical Forms)

1. *If* $\Gamma \vdash \mathsf{M} : \mathsf{A}$ *then there exists a canonical object* $\bar{\bar{\mathsf{M}}}$ *such that* $\Gamma \vdash \bar{\bar{\mathsf{M}}} : \mathsf{A}$ *and* $\Gamma \vdash \mathsf{M} = \bar{\bar{\mathsf{M}}} : \mathsf{A}$.
2. *If* $\Gamma \vdash \mathsf{A} : \mathsf{K}$ *then there exists a canonical family* $\bar{\bar{\mathsf{A}}}$ *such that* $\Gamma \vdash \bar{\bar{\mathsf{A}}} : \mathsf{K}$ *and* $\Gamma \vdash \mathsf{A} = \bar{\bar{\mathsf{A}}} : \mathsf{K}$.
3. *If* $\Gamma \vdash \mathsf{K} : \mathsf{kind}$ *then there exists a canonical kind* $\bar{\mathsf{K}}$ *such that* $\Gamma \vdash \bar{\mathsf{K}} : \mathsf{kind}$ *and* $\Gamma \vdash \mathsf{K} = \bar{\mathsf{K}} : \mathsf{kind}$.

Proof

Direct, from Canonical Mediating Terms for Equality Algorithm. We will show the case for objects.

$\Gamma \vdash \mathsf{M} = \mathsf{M} : \mathsf{A}$ By Reflexivity
$\Gamma^- \vdash \mathsf{M} \Longleftrightarrow \mathsf{M} : \mathsf{A}^-$ By Completeness of Algorithm
There exists a canonical object $\bar{\bar{\mathsf{M}}}$ such that $\Gamma \vdash \bar{\bar{\mathsf{M}}} : \mathsf{A}$,
$\Gamma \vdash \mathsf{M} = \bar{\bar{\mathsf{M}}} : \mathsf{A}$ By Canonical Mediating Terms for Equality Algorithm

□

9 Decidability of Type Checking

Terms which are related by the algorithm for equality are called by Harper and Pfenning "normalizing". This terminology is justified by our result on canonical forms, since these terms are provably equal to canonical forms. We first prove that equality between normalizing terms is decidable. This will imply decidability of equality for all well-typed terms, by the use of completeness of the algorithm for deciding equality.

Lemma 9.1 (Decidability of Equality for Normalizing Terms)

1. *If* $\Delta \vdash \mathsf{M}_1 \Longleftrightarrow \mathsf{M}_2 : \tau$ *and* $\Delta \vdash \mathsf{M}_3 \Longleftrightarrow \mathsf{M}_4 : \tau$ *then it is decidable whether* $\Delta \vdash \mathsf{M}_1 \Longleftrightarrow \mathsf{M}_3 : \tau$.
2. *If* $\Delta \vdash \mathsf{M}_1 \longleftrightarrow \mathsf{M}_2 : \tau_1$ *and* $\Delta \vdash \mathsf{M}_3 \longleftrightarrow \mathsf{M}_4 : \tau_2$ *then it is decidable whether* $\Delta \vdash \mathsf{M}_1 \longleftrightarrow \mathsf{M}_3 : \tau_3$ *for any* τ_3.
3. *If* $\Delta \vdash \mathsf{A}_1 \Longleftrightarrow \mathsf{A}_2 : \kappa$ *and* $\Delta \vdash \mathsf{A}_3 \Longleftrightarrow \mathsf{A}_4 : \kappa$ *then it is decidable whether* $\Delta \vdash \mathsf{A}_1 \Longleftrightarrow \mathsf{A}_3 : \kappa$.
4. *If* $\Delta \vdash \mathsf{A}_1 \longleftrightarrow \mathsf{A}_2 : \kappa_1$ *and* $\Delta \vdash \mathsf{A}_3 \longleftrightarrow \mathsf{A}_4 : \kappa_2$ *then it is decidable whether* $\Delta \vdash \mathsf{A}_1 \longleftrightarrow \mathsf{A}_3 : \kappa_3$ *for any* κ_3.
5. *If* $\Delta \vdash \mathsf{K}_1 \longleftrightarrow \mathsf{K}_2 : \mathsf{kind}^-$ *and* $\Delta \vdash \mathsf{K}_3 \longleftrightarrow \mathsf{K}_4 : \mathsf{kind}^-$ *then it is decidable whether* $\Delta \vdash \mathsf{K}_1 \longleftrightarrow \mathsf{K}_3 : \mathsf{kind}^-$.

Proof

By structural induction on the derivation of the judgment. □

Theorem 9.2 (Decidability of Algorithmic Equality)

1. *If* $\Gamma \vdash \mathsf{M}_1 : \mathsf{A}$ *and* $\Gamma \vdash \mathsf{M}_2 : \mathsf{A}$, *then it is decidable whether* $\Gamma^- \vdash \mathsf{M}_1 \Longleftrightarrow \mathsf{M}_2 : \mathsf{A}^-$.
2. *If* $\Gamma \vdash \mathsf{A}_1 : \mathsf{K}$ *and* $\Gamma \vdash \mathsf{A}_2 : \mathsf{K}$, *then it is decidable whether* $\Gamma^- \vdash \mathsf{A}_1 \Longleftrightarrow \mathsf{A}_2 : \mathsf{K}^-$.
3. *If* $\Gamma \vdash \mathsf{K}_1 : \mathsf{kind}$ *and* $\Gamma \vdash \mathsf{K}_2 : \mathsf{kind}$, *then it is decidable whether* $\Gamma^- \vdash \mathsf{K}_1 \longleftrightarrow \mathsf{K}_2 : \mathsf{kind}^-$.

Proof

Direct from decidability for normalizing terms, using reflexivity of equality. We show one case, others are analogous.

Case 1:

$\Gamma \vdash \mathsf{M}_1 = \mathsf{M}_1 : \mathsf{A}$	By Reflexivity of Equality
$\Gamma^- \vdash \mathsf{M}_1 \Longleftrightarrow \mathsf{M}_1 : \mathsf{A}^-$	By Completeness of Algorithmic Equality
$\Gamma \vdash \mathsf{M}_2 = \mathsf{M}_2 : \mathsf{A}$	By Reflexivity of Equality
$\Gamma^- \vdash \mathsf{M}_2 \Longleftrightarrow \mathsf{M}_2 : \mathsf{A}^-$	By Completeness of Algorithmic Equality
It is decidable whether $\Gamma^- \vdash \mathsf{M}_1 \Longleftrightarrow \mathsf{M}_2 : \mathsf{A}^-$	By Decidability of Algorithmic Equality for Normalizing Terms

□

Corollary 9.3 (Decidability of Definitional Equality)

1. *If* $\Gamma \vdash \mathsf{M}_1 : \mathsf{A}$ *and* $\Gamma \vdash \mathsf{M}_2 : \mathsf{A}$, *then it is decidable whether* $\Gamma^- \vdash \mathsf{M}_1 = \mathsf{M}_2 : \mathsf{A}^-$.
2. *If* $\Gamma \vdash \mathsf{A}_1 : \mathsf{K}$ *and* $\Gamma \vdash \mathsf{A}_2 : \mathsf{K}$, *then it is decidable whether* $\Gamma^- \vdash \mathsf{A}_1 = \mathsf{A}_2 : \mathsf{K}^-$.
3. *If* $\Gamma \vdash \mathsf{K}_1 : \mathsf{kind}$ *and* $\Gamma \vdash \mathsf{K}_2 : \mathsf{kind}$, *then it is decidable whether* $\Gamma^- \vdash \mathsf{K}_1 = \mathsf{K}_2 : \mathsf{kind}$.

Proof

By soundness and completeness of algorithmic equality, it is enough to have decidability of algorithmic equality. □

9.1 Algorithm for Type Checking

We now use the algorithm for equality as a subroutine to provide an algorithm for type checking.

Objects

$$\frac{\Gamma(\mathsf{x}) = \mathsf{A}}{\Gamma \vdash \mathsf{x} \Rightarrow \mathsf{A}} \qquad \frac{\Sigma(\mathsf{c}) = \mathsf{A}}{\Gamma \vdash \mathsf{c} \Rightarrow \mathsf{A}}$$

$$\frac{\Gamma \vdash \mathsf{M}_1 \Rightarrow \mathsf{A} \qquad \mathsf{A} \stackrel{\text{whr}}{\longrightarrow}^* \Pi\mathsf{x}{:}\mathsf{A}_{21}.\mathsf{A}_1 \qquad \Gamma \vdash \mathsf{M}_2 \Rightarrow \mathsf{A}_{22} \qquad \Gamma^- \vdash \mathsf{A}_{21} \Longleftrightarrow \mathsf{A}_{22} : \mathsf{type}^-}{\Gamma \vdash \mathsf{M}_1\, \mathsf{M}_2 \Rightarrow \mathsf{A}_1\, [\mathsf{M}_2/\mathsf{x}]}$$

$$\frac{\Gamma \vdash \mathsf{A}_1 \Rightarrow \mathsf{type} \qquad \Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{M}_2 \Rightarrow \mathsf{A}_2}{\Gamma \vdash \lambda\mathsf{x}{:}\mathsf{A}_1.\mathsf{M}_2 \Rightarrow \Pi\mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2}$$

$$\frac{\begin{array}{c}\Gamma \vdash \Sigma\mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 \Rightarrow \mathsf{type} \qquad \Gamma \vdash \mathsf{M}_1 \Rightarrow \mathsf{A}_{11} \\ \Gamma^- \vdash \mathsf{A}_{11} \Longleftrightarrow \mathsf{A}_1 : \mathsf{type}^- \qquad \Gamma \vdash \mathsf{M}_2 \Rightarrow \mathsf{A}_{21} \\ \Gamma^- \vdash \mathsf{A} \Longleftrightarrow \mathsf{A}_2\, [\mathsf{M}_1/\mathsf{x}] : \mathsf{type}^-\end{array}}{\Gamma \vdash \langle \mathsf{M}_1, \mathsf{M}_2 \rangle^{\Sigma\mathsf{x}:\mathsf{A}_1.\mathsf{A}_2} \Rightarrow \Sigma\mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2}$$

$$\frac{\Gamma \vdash \mathsf{M} \Rightarrow \mathsf{A} \qquad \mathsf{A} \stackrel{\text{whr}}{\longrightarrow}^* \Sigma\mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2}{\Gamma \vdash \pi_1\mathsf{M} \Rightarrow \mathsf{A}_1} \qquad \frac{\Gamma \vdash \mathsf{M} \Rightarrow \mathsf{A} \qquad \mathsf{A} \stackrel{\text{whr}}{\longrightarrow}^* \Sigma\mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2}{\Gamma \vdash \pi_2\mathsf{M} \Rightarrow \mathsf{A}_2\, [\pi_1\mathsf{M}/\mathsf{x}]} \qquad \frac{}{\Gamma \vdash \langle\rangle \Rightarrow 1}$$

Families

$$\frac{\Sigma(\mathsf{a}) = \mathsf{K}}{\Gamma \vdash \mathsf{a} \Rightarrow \mathsf{K}} \qquad \frac{\Gamma \vdash \mathsf{A}_1 \Rightarrow \mathsf{type} \quad \Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{A}_2 \Rightarrow \mathsf{K}}{\Gamma \vdash \lambda\mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 \Rightarrow \Pi\mathsf{x}{:}\mathsf{A}_1.\mathsf{K}}$$

$$\frac{\Gamma \vdash \mathsf{A}_1 \Rightarrow \Pi\mathsf{x}{:}\mathsf{A}_{21}.\mathsf{K}_1 \qquad \Gamma \vdash \mathsf{M} \Rightarrow \mathsf{A}_{22} \qquad \Gamma^- \vdash \mathsf{A}_{21} \Longleftrightarrow \mathsf{A}_{22} : \mathsf{K}_1{}^-}{\Gamma \vdash \mathsf{A}_1\, \mathsf{M} \Rightarrow \mathsf{K}_1\, [\mathsf{M}/\mathsf{x}]}$$

$$\frac{\Gamma \vdash \mathsf{A}_1 \Rightarrow \mathsf{type} \qquad \Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{A}_2 \Rightarrow \mathsf{type}}{\Gamma \vdash \Pi\mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 \Rightarrow \mathsf{type}} \qquad \frac{\Gamma \vdash \mathsf{A}_1 \Rightarrow \mathsf{type} \qquad \Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{A}_2 \Rightarrow \mathsf{type}}{\Gamma \vdash \Sigma\mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 \Rightarrow \mathsf{type}}$$

$$\frac{}{\Gamma \vdash 1 \Rightarrow \mathsf{type}}$$

Kinds

$$\frac{}{\Gamma \vdash \mathsf{type} \Rightarrow \mathsf{kind}} \qquad \frac{\Gamma \vdash \mathsf{A} \Rightarrow \mathsf{type} \qquad \Gamma, \mathsf{x}{:}\mathsf{A} \vdash \mathsf{K} \Rightarrow \mathsf{kind}}{\Gamma \vdash \Pi\mathsf{x}{:}\mathsf{A}.\mathsf{K} \Rightarrow \mathsf{kind}}$$

It is easy to show the soundness and completeness of the algorithm for typechecking.

Lemma 9.4 (Soundness of Algorithmic Typechecking)

1. *If* $\Gamma \vdash \mathsf{M} \Rightarrow \mathsf{A}$ *then* $\Gamma \vdash \mathsf{M} : \mathsf{A}$.
2. *If* $\Gamma \vdash \mathsf{A} \Rightarrow \mathsf{K}$ *then* $\Gamma \vdash \mathsf{A} : \mathsf{K}$.
3. *If* $\Gamma \vdash \mathsf{K} \Rightarrow \mathsf{kind}$ *then* $\Gamma \vdash \mathsf{K} : \mathsf{kind}$.

Proof
By induction on the derivation.

□

To show completeness under the presence of a non-trivial equality at the type level (induced by the presence of type-level abstractions), we need the following technical lemma.

Lemma 9.5 (Inversion on Product and Sum Families)

1. *If* $\Gamma \vdash \mathsf{A} = \Pi \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 : \mathsf{type}$ *then* $\mathsf{A} \stackrel{\text{whr}}{\longrightarrow}^* \Pi \mathsf{x}{:}\mathsf{A}_{11}.\mathsf{A}_{21}$ *such that* $\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_{11} : \mathsf{type}$ *and* $\Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{A}_2 = \mathsf{A}_{21} : \mathsf{type}$.

2. *If* $\Gamma \vdash \mathsf{A} = \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 : \mathsf{type}$ *then* $\mathsf{A} \stackrel{\text{whr}}{\longrightarrow}^* \Sigma \mathsf{x}{:}\mathsf{A}_{11}.\mathsf{A}_{21}$ *such that* $\Gamma \vdash \mathsf{A}_1 = \mathsf{A}_{11} : \mathsf{type}$ *and* $\Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{A}_2 = \mathsf{A}_{21} : \mathsf{type}$.

Proof
By Erasure Preservation under Equality together with Inversion on Erasure, and using Injectivity.

□

Lemma 9.6 (Completeness of Algorithmic Typechecking)

1. *If* $\Gamma \vdash \mathsf{M} : \mathsf{A}$ *then* $\Gamma \vdash \mathsf{M} \Rightarrow \mathsf{A}'$ *and* $\Gamma \vdash \mathsf{A} = \mathsf{A}' : \mathsf{type}$.

2. *If* $\Gamma \vdash \mathsf{A} : \mathsf{K}$ *then* $\Gamma \vdash \mathsf{A} \Rightarrow \mathsf{K}'$ *and* $\Gamma \vdash \mathsf{K} = \mathsf{K}' : \mathsf{kind}$.

3. *If* $\Gamma \vdash \mathsf{K} : \mathsf{kind}$ *then* $\Gamma \vdash \mathsf{K} \Rightarrow \mathsf{kind}$.

Proof
By induction on the derivation. We show two significant cases.

Case 1: $$\frac{\Gamma \vdash \mathsf{M}_1 : \Pi \mathsf{x}{:}\mathsf{A}_2.\mathsf{A}_1 \qquad \Gamma \vdash \mathsf{M}_2 : \mathsf{A}_2}{\Gamma \vdash \mathsf{M}_1\, \mathsf{M}_2 : \mathsf{A}_1\, [\mathsf{M}_2/\mathsf{x}]}$$

$\Gamma \vdash \mathsf{M}_1 \Rightarrow \mathsf{A}_{11}$,
$\Gamma \vdash \mathsf{A}_{11} = \Pi \mathsf{x}{:}\mathsf{A}_2.\mathsf{A}_1 : \mathsf{type}$ — By induction
$\mathsf{A}_{11} \stackrel{\text{whr}}{\longrightarrow}^* \Pi \mathsf{x}{:}\mathsf{A}'_2.\mathsf{A}'_1$ $\Gamma \vdash \mathsf{A}_2 = \mathsf{A}'_2 : \mathsf{type}$,
$\Gamma, \mathsf{x}{:}\mathsf{A}_2 \vdash \mathsf{A}_1 = \mathsf{A}'_1 : \mathsf{type}$ — By Inversion on Product Families
$\Gamma \vdash \mathsf{M}_2 \Rightarrow \mathsf{A}_{21}$,
$\Gamma \vdash \mathsf{A}_{21} = \mathsf{A}_2 : \mathsf{type}$ — By induction
$\Gamma \vdash \mathsf{A}_{21} = \mathsf{A}'_2 : \mathsf{type}$ — By rule (transitivity)
$\Gamma^- \vdash \mathsf{A}_{21} \Longleftrightarrow \mathsf{A}'_2 : \mathsf{type}^-$ — By Completeness of Algorithmic Equality
$\Gamma \vdash \mathsf{M}_1\, \mathsf{M}_2 \Rightarrow \mathsf{A}'_1\, [\mathsf{M}_2/\mathsf{x}]$ — By rule
$\Gamma \vdash \mathsf{A}'_1\, [\mathsf{M}_2/\mathsf{x}] = \mathsf{A}_1\, [\mathsf{M}_2/\mathsf{x}] : \mathsf{type}$ — By Functionality

Case 2: $$\frac{\Gamma \vdash \mathsf{M} : \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2}{\Gamma \vdash \pi_1 \mathsf{M} : \mathsf{A}_1}$$

$\Gamma \vdash \mathsf{M} \Rightarrow \mathsf{A}$,
$\Gamma \vdash \mathsf{A} = \Sigma \mathsf{x}{:}\mathsf{A}_1.\mathsf{A}_2 : \mathsf{type}$ — By induction
$\mathsf{A} \stackrel{\text{whr}}{\longrightarrow}^* \Sigma \mathsf{x}{:}\mathsf{A}'_1.\mathsf{A}'_2$,
$\Gamma \vdash \mathsf{A}_1 = \mathsf{A}'_1 : \mathsf{type}$,
$\Gamma, \mathsf{x}{:}\mathsf{A}_1 \vdash \mathsf{A}_2 = \mathsf{A}'_2 : \mathsf{type}$ — By Inversion on Sum Families
$\Gamma \vdash \pi_1 \mathsf{M} \Rightarrow \mathsf{A}'_1$ — By rule

□

10 Conclusion

We have presented a new type theory in this paper. This is the logical framework LF extended with dependent pair and unit types. To be used as a logical framework, the important property of canonical forms is essential. We have shown the existence of the canonical forms property. The other useful property of decidability of type checking and equivalence is also shown. The algorithm is an extension of previous algorithms for the restriction to LF. A careful proof of soundness and completeness has also been shown.

In future work, we intend to use this language as our representation language for a functional programming language that can express soundness of algorithms. The current results will be important in showing that such a language can be verified statically.

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